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COMPLEMENTS OF POTENTIAL THEORY.*

By GRIFFITH C. EVANS.

Introduction. Let C be a simple, closed, rectifiable plane curve, and T_+, T_- the interior and exterior regions, respectively, bounded by the curve. Let Q be a fixed and P a variable point on C, and let $\nu(e)$ be an additive function of point sets e on C, defined for all sets e, measurable in the Borel sense, with respect to arc length s of C. We are to consider the limiting values of potentials of a double layer, and the derivatives of potentials of a single layer, formed for these general mass distributions $\nu(e)$, as the point M at which they are given approaches Q from T_+ or from T_-.

In order to obtain these limiting values a restriction on the nature of the curve in the neighborhood of the particular point Q is unavoidable (see § 2, below). It is essentially a smoothing of the curve which is necessary, rather than a smoothing of the mass distribution, as the examples which are to be given will show.

The coördinates x, y of P are absolutely continuous functions of s, and the quantities $\cos(x, s)$, $\cos(y, s)$ are defined almost everywhere on C, and are bounded, in numerical value ≤ 1. Moreover, irrespective of their definition on the remaining set of zero measure, they are summable, and the formulae †

$$x_{P_2} - x_{P_1} = \int_{P_1}^{P_2} \cos(x, s)\, ds, \qquad y_{P_2} - y_{P_1} = \int_{P_1}^{P_2} \cos(y, s)\, ds$$

are valid, for any two points P_1, P_2 on C.

LEMMA I. *Let $P_1, P_2, \cdots$ be an infinite sequence of points, lying ahead of P on C, with $\lim P_k = P$, and suppose that the direction of the ray PP_k has a corresponding limit which makes an angle θ with a fixed direction. If for some neighborhood of P (exclusive of P), measured from P in the positive sense on C, the direction θ' of the curve, wherever it exists, satisfies*

* Presented in part to the American Mathematical Society, September, 1931, under the title: "The Cauchy Integral and Summable Boundary Values." The part of Theorem I, below, relating to $u(M)$, $u_1(M)$ has been proved for surfaces in Evans and Miles, "Potentials of General Masses in Single and Double Layers," *American Journal of Mathematics*, Vol. 53 (1931), pp. 493-516, but only in the case where approach is along the normal to the surface.

† Lebesgue, *Leçons sur l'intégration*, Paris (1928), pp. 198-201.

the inequality $\theta_1 < \theta' < \theta_2$, *with* $\theta_2 - \theta_1 \leq \pi$ *then* θ *satisfies the inequality* $\theta_1 \leq \theta \leq \theta_2$.

We take $\theta_1 = 0$, and that direction as the axis of x; and we take the origin at P. We assume first that $\theta_2 \leq \pi/2$. We have, almost everywhere,

$$0 < \cos(y, s) < \sin\theta_2 \leq 1, \quad 0 \leq \cos\theta_2 < \cos(x, s) < 1, \quad \text{for} \quad 0 < s \leq \delta,$$

and from the integral formulae given above,

$$0 < y_k < s_k \sin\theta_2, \qquad x_k > s_k \cos\theta_2 \geq 0, \qquad 0 < y_k/x_k < \tan\theta_2,$$

from which the truth of the Lemma is evident.

Suppose finally that $\pi/2 < \theta_2 \leq \pi$. We still have $y_k > 0$, and therefore $\theta \geq 0$. If $\theta \leq \pi/2$ the proposition is verified. We suppose then that from a certain k we have $\theta_k > \pi/2$. Let s'_k denote the measure of the portion of the arc PP_k for which $\cos(x, s)$, where it exists, is negative. Then

$$y_k > s'_k \sin\theta_2 \geq 0, \qquad |x_k| < s'_k |\cos\theta_2|, \qquad y_k/|x_k| > |\tan\theta_2| \geq 0,$$

from which the truth of the Lemma is again evident. We cannot however extend the proposition to the case where $\theta_2 - \theta_1 > \pi$. A similar statement holds of course for approach to P along C in the positive sense.

If we are dealing with, say, the potential of a double layer due to an absolutely continuous mass function—that is, a density distribution—on C, it is unnecessary to say anything about the points where a direction of the curve fails to exist; for these constitute a set of measure zero on C, and the integral which gives the potential is not affected by changing the doublets on this set. On the other hand, if we have a general distribution of mass as a double layer, it may even happen that a doublet of positive mass will be placed at some single point of the curve where there is no tangent direction, and the value of the potential will involve a not-zero term depending on the orientation of the doublet at this point. We must therefore, in specifying a distribution of a double layer, define an orientation of the doublet at every point of C—at least, at every point where the density is infinite—whether or not the curve itself has a direction at that point. We do this by assigning an approximate direction to the curve.

Let P be an arbitrary point of C, and P' a point nearby at a distance ϵ from P in the positive sense around C; let θ_ϵ be the angle made by PP' with a fixed x-direction. For a given ϵ, θ_ϵ is a continuous function of P, and the superior and inferior limits

$$\overline{\lim_{\epsilon=0}}\ \theta_\epsilon = \bar{\theta}(P), \qquad \underline{\lim_{\epsilon=0}}\ \theta_\epsilon = \underline{\theta}(P)$$

are functions measurable in the Borel sense on C.

The set of points where one of the quantities $\bar{\theta}(P)$, $\underline{\theta}(P)$ is infinite, is measurable in the Borel sense, and therefore the following function is likewise measurable in that sense:

$$\theta_+(P) = (1/2)[\bar{\theta}(P) + \underline{\theta}(P)] - 2n(P)\pi, \quad \text{if } \bar{\theta}(P),\ \underline{\theta}(P) \text{ are finite,}$$
$$n(P) \text{ an integer, chosen so that } 0 \leq \theta_+(P) < 2\pi,$$
$$= 0, \quad \text{if } \bar{\theta}(P) \text{ or } \underline{\theta}(P) \text{ is not finite.}$$

Similarly a limiting direction $\theta_-(P)$ of $P'P$ is chosen, for approach of P' to P in a positive sense along C. The function

$$\theta(P) = (1/2)[\theta_+(P) + \theta_-(P)]$$

is then bounded, measurable in the Borel sense, and $\tan\theta = dy/dx$, $\cos\theta = dx/ds$, $\sin\theta = dy/ds$, wherever these quantities exist and satisfy the condition $(dx/ds)^2 + (dy/ds)^2 = 1$.

For convenience, we call the direction indicated by $\theta(P)$ the direction of C at P, and fix the direction of the doublet at P in the customary way with reference to this direction. Definitions which make use of averages would of course also be possible, but they are not necessary here.

Let M be a point of T_+ or T_-, $r = MP$ (with direction MP when angles are to be specified), n_P the direction of the normal at P, taken towards T_+, and perpendicular to the direction, as just defined, of the curve at P. Then the quantities

$$\cos\phi = \cos(r, n_P) = \cos(x, n_P)\cos(x, r) + \cos(y, n_P)\cos(y, r)$$
$$\sin\phi = \sin(r, n_P) = \cos(r, s_P)$$

are everywhere defined on C, measurable in the Borel sense, and in absolute value ≤ 1. The quantities

$$\cos\phi' = \cos(r, n_Q), \qquad \sin\phi' = \sin(r, n_Q)$$

with a fixed Q, are defined for all P on C, are continuous, and in numerical value ≤ 1.

The four Stieltjes integrals

(A) $$u(M) = \int_C r^{-1}\cos\phi\, dv(e_P)$$

(B) $$u_1(M) = \int_C r^{-1}\cos\phi'\, dv(e_P)$$

(C) $$u_2(M) = \int_C r^{-1}\sin\phi\, dv(e_P)$$

(D) $$u_3(M) = \int_C r^{-1}\sin\phi'\, dv(e_P)$$

are now all defined for M in T_+ or in T_-. The integrals (B) and (D) are given in the usual sense; those in (A) and (C) are Radon * or Daniell † generalized integrals. The (A) and (C) may be regarded as potentials due to general mass distributions of doublets, respectively normal and tangent to the curve C; the integrals (B) and (D) are the derivatives in the fixed direction n_Q and perpendicular to it, respectively, of a general simple mass distribution on C.

A general mass distribution of doublets along C, whose orientation is determined at points of C by direction cosines which are measurable in the Borel sense with respect to s, may be replaced, as far as its potential is concerned, by two integrals of forms (A) and (C), with different mass functions $\nu_1(e)$, $\nu_2(e)$ respectively. For if $h(P)$ is bounded and measurable (B) the integral

$$\int_Q^P h(P')\,d\nu(e_{P'})$$

is a function of bounded variation on C and determines such a distribution of mass.

In conclusion to these introductory considerations, we introduce the following condition (γ) with respect to the point Q. We make the convention that the symbol s merely denotes numerical value, except when, in conjunction with some other letter, it forms part of the symbol for an angle.

(γ). *Given the fixed point Q and the variable point P, on C, there subsists the inequality* $| \measuredangle (n_Q, n_P) | < \gamma(s)$, *$s$ the arc QP, where $\gamma(s)$, for $s > 0$, is a positive, monotonic (non-decreasing) function, such that the integral*

$$\int_0^s \gamma(s)/s\,ds = m(s)$$

is convergent.

We have $\lim m(s) = 0$ if $\lim s = 0$.

It is not necessary to assume that $\gamma(s)$ is continuous. Moreover, if the condition (γ) is assumed to hold merely for points P in the neighborhood of Q where dx/ds and dy/ds are defined in the usual way, it will hold throughout, with the extended definition of the direction of C, provided that we write $\gamma(s) = \gamma(s + 0)$. For that is the statement expressed in Lemma I.

---

* J. Radon, "Theorie und Anwendung der absolut additiven Mengenfunktionen," *Sitzungsberichte der Akademie der Wissenschaften*, Wien (1913), p. 1295.

† P. J. Daniell, (I) "A General Form of Integral," *Annals of Mathematics*, Vol. 19 (1918), pp. 279-294; (II) "Further Properties of the General Integral," *ibid.*, Vol. 21 (1920), pp. 203-220.

The following theorem will be proved.

THEOREM I. *Let the condition* (γ) *be satisfied at the point* Q *of* C, *and let* $\nu(e)$ *have at* Q *a unique derivative* * $\nu'(e) = A$, *with respect to* s. *Denote by* M' *the image of* M *by reflexion across the tangent line at* Q. *Then as* M *in* T_+ *approaches* Q ($\lim_{M=Q+}$) *or as* M *in* T_- *approaches* Q ($\lim_{M=Q-}$), *in the wide sense,*† *the following relations hold*

$$(E) \qquad \lim_{M=Q\pm} u(M) = \mp 2\pi A + \int_C \frac{\cos(QP, n_P)}{QP}\, d\nu(e_P)$$

$$(E_1) \qquad \lim_{M=Q\pm} u_1(M) = \mp 2\pi A + \int_C \frac{\cos(QP, n_Q)}{QP}\, d\nu(e_P)$$

$$(E_2) \qquad \lim_{M=Q\pm} \{u_2(M) - u_2(M')\} = 0$$

$$(E_3) \qquad \lim_{M=Q\pm} \{u_3(M) - u_3(M')\} = 0.$$

1. *Lemmas on Stieltjes integrals.* Although special cases of the following proposition are frequently used, it is convenient to have the statement in explicit form. The integrals are generalized Stieltjes integrals.

LEMMA II. *Let* $f(x)$ *be bounded and measurable in the Borel sense, and* $g_1(x)$, $g_2(x)$ *of bounded variation,* $a \leq x \leq b$; *then* $g(x) = g_1(x)g_2(x)$ *is of bounded variation, and*

$$(1.1) \qquad \int_a^b f(x)\,dg(x) = \int_a^b f(x)g_1(x)\,dg_2(x) + \int_a^b f(x)g_2(x)\,dg_1(x).$$

The boundedness of the variation of $g(x)$ is of course well known. It is a consequence of the identity

$$g(x_{i+1}) - g(x_i) = g_1(x_{i+1})\{g_2(x_{i+1}) - g_2(x_i)\} + g_2(x_i)\{g_1(x_{i+1}) - g_1(x_i)\}.$$

When $f(x)$, $g_1(x)$, $g_2(x)$ are, in addition, all continuous, (1.1) follows immediately from the definition of the Riemann-Stieltjes integral. In fact

$$\Sigma f(x_i)\{g(x_{i+1}) - g(x_i)\} = \Sigma f(x_i)g_1(x_{i+1})\{g_2(x_{i+1}) - g_2(x_i)\} + \Sigma f(x_i)g_2(x_i)\{g_1(x_{i+1}) - g_1(x_i)\}.$$

---

* That is, a unique derivative with respect to regular families.

† Let Q be a point of C and Δ_1 a circular sector with vertex at Q, but otherwise contained, with its boundary, in T_+ (T_-). Let Δ_2 be a circular sector with vertex at Q, but otherwise contained, with its boundary, in the interior of Δ_1. We say that M approaches Q in the wide sense from T_+ (T_-) if, approaching Q, it comes and remains eventually within some Δ_2.

The lemma is then established by passages to the limit with (1.1) for continuous functions as a basis.

For this purpose we write $g_{nm}(x) = g_{1n}(x)g_{2m}(x)$ where $g_{ik}(x)$, $(i = 1, 2)$, is constructed so as to have the following properties:

(a) $g_{ik}(x)$ is continuous, $a \leq x \leq b$.

(b) $\lim_{k=\infty} g_{ik}(x) = g_i(x)$, $a \leq x \leq b$.

(c) $| g_{ik}(x_2) - g_{ik}(x_1) | \leq t(x_2) - t(x_1)$, $a \leq x_1 < x_2 \leq b$, where $t(x)$ is of bounded variation.

The property (c) implies the following:

(d) $g_{ik}(x)$ is of bounded variation uniformly for all k.

This construction can be made by a polygonal approximation.

Consider for instance a function of bounded variation $\alpha(x) = p(x) - n(x) + \alpha(a)$ where $p(x)$ and $-n(x)$ are respectively positive and negative variation functions for $\alpha(x)$. For $\alpha_m(x)$ we take $p_m(x) - n_m(x)$, where $p_m(x)$ is a polygonal function, formed by joining successively with straight lines a finite number of points on the graph of $p(x)$, the vertices for $p_{m+1}(x)$ to be obtained by adding new ones to those of $p_m(x)$ in such a way that as m tends to ∞ they eventually become dense in (a, b) and include the points for which $x = a$ and $x = b$, and all the points of discontinuity of $p(x)$. A similar construction is used for $n_m(x)$. Evidently then $\lim\ (m = \infty)\ \alpha_m(x) = \alpha(x)$, $\alpha_m(x)$ is continuous, and

$$| \alpha_m(x_2) - \alpha_m(x_1) | \leq t(x_2) - t(x_1)$$

where $t(x)$ is the total variation function of $\alpha(x)$, namely $p(x) + n(x)$.

Let $f(x)$ remain continuous. Equation (1.1) holds when $g_1(x)$, $g_2(x)$ and $g(x)$ are replaced respectively by $g_{1n}(x)$, $g_{2m}(x)$ and $g_{nm}(x)$. From (a), (b), (d) it follows, as a well known theorem,* that

$$\lim_{n=\infty} \int_a^b f(x)g_{2m}(x)\,dg_{1n}(x) = \int_a^b f(x)g_{2m}(x)\,dg_1(x)$$

$$\lim_{n=\infty} \int_a^b f(x)\,dg_{nm}(x) = \int_a^b f(x)\,dg_m(x)$$

where $g_m(x) = \lim\ (n = \infty)\ g_{nm}(x)$ is of bounded variation, uniformly in m, and $\lim\ (m = \infty)\ g_m(x) = g(x)$. Also

* Helly, *Sitzungsberichte der Akademie der Wissenschaften*, Wien (1912), p. 288; Radon, *ibid., loc. cit.*; Bray, *Annals of Mathematics*, Vol. 20 (1919), p. 180.

$$\lim_{n=\infty} \int_a^b f(x) g_{1n}(x) dg_{2m}(x) = \int_a^b f(x) g_1(x) dg_{2m}(x),$$

as a fundamental property of the general integral. Accordingly (1. 1) holds when $f(x)$ is continuous, and $g_2(x)$ is replaced by $g_{2m}(x)$.

Let now m tend to ∞. We have, as above,

$$\lim_{m=\infty} \int_a^b f(x) dg_m(x) = \int_a^b f(x) dg(x)$$

$$\lim_{m=\infty} \int_a^b f(x) g_{2m}(x) dg_1(x) = \int_a^b f(x) g_2(x) dg_1(x),$$

and an extension by Daniell * of the theorem, referred to above, yields the fact that

$$\lim_{m=\infty} \int_a^b f(x) g_1(x) dg_{2m}(x) = \int_a^b f(x) g_1(x) dg_2(x).$$

Consequently (1. 1) holds, under the hypotheses of the lemma, provided that $f(x)$ is continuous. That it is also valid when $f(x)$ is merely bounded and measurable in the Borel sense is an immediate consequence of the postulates which define the general integral.†

Given a function $\nu(x)$, of bounded variation, we say that it has a unique derivative, on regular families, at $x = 0$, if the corresponding function of plurisegments has a unique derivative, on regular families, at $x = 0$, or, what amounts to the same thing, if it is continuous at $x = 0$, and if the corresponding additive function of point sets has such a derivative. A plurisegment r is the sum of a finite number or of a denumerable infinity of segments s, $s = (x', x'')$, no two of the segments having interior points in common; the measure of a plurisegment is the sum of the measures of its segments. We say that $\nu(r)$ corresponds to $\nu(x)$ if when s is the interval (x', x''), $\nu(s) = \nu(x'') - \nu(x')$. The theory of such functions is given by Vitali ‡ and Maria.§

---

* Daniell, *loc. cit.* (II), see 3(5a), p. 218. If $\lim (m = \infty)\ \alpha_m(x) = \alpha(x)$, $a \leq x \leq b$ and $|\alpha_m(x_2) - \alpha_m(x_1)| \leq \beta(x_2) - \beta(x_1)$ for all x_1, x_2, $a \leq x_1 < x_2 \leq b$, independently of m where $\beta(x)$ is of bounded variation, then

$$\lim_{m=\infty} \int_a^b f(x) d\alpha_m(x) = \int_a^b f(x) d\alpha(x),$$

provided that $f(x)$ is summable with respect to $\beta(x)$.

† Daniell, *loc. cit.* (I).

‡ G. Vitali, "Analisi delle funzioni a variazione limitata," *Rendiconti del Circolo Matematico di Palermo*, Vol. 46 (1922), p. 388.

§ A. Maria, "Functions of Plurisegments," *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 448-471.

A sequence of plurisegments r_i forms a regular family about $x = 0$, with parameter of regularity k, if

$$\lim_{i=\infty} \frac{\text{meas. } r_i}{\omega_i} = k > 0.$$

where ω_i is the measure of the smallest segment with center $x = 0$ which contains r_i. The function $\nu(r)$ has a unique derivative A, on regular families, at $x = 0$, if

$$\lim_{i=\infty} \frac{f(r_i)}{\text{meas. } r_i} = A$$

for every regular family $\{r_i\}$ about $x = 0$.

The following lemma * will accordingly be proved if we prove the corresponding theorem for additive functions of plurisegments. It is also true when stated in terms of additive functions of point sets.

LEMMA III. *If $\nu(x)$ has the unique derivative A (on regular families) at $x = 0$, the total variation function $t(x)$ of $\nu(x)$ has the unique derivative $|A|$ at $x = 0$; moreover the positive variation function $p(x)$ has the unique derivative $\{|A| + A\}/2$, and the negative variation function $-n(x)$ has the unique derivative $-\{|A| - A\}/2$, at $x = 0$.*

Let r_i constitute a regular family of plurisegments about $x = 0$, with parameter of regularity k. We consider first $A > 0$, and show that

$$\lim (i = \infty)\ p(r_i)/\text{meas. } r_i = A.$$

Given r_i we can find a finite plurisegment f_i, composed of a finite number of the segments of r_i, which has the following properties:

$$\text{meas. } (r_i - f_i) < \delta_i, \quad \delta_i \text{ arbitrary, } > 0,$$
$$p(r_i) - p(f_i) < \eta_i, \quad \eta_i \text{ arbitrary, } > 0,$$
$$\lim_{i=\infty} \frac{\text{meas. } f_i}{\omega_i} = k,$$
$$\lim_{i=\infty} \left| \frac{p(r_i)}{\text{meas. } r_i} - \frac{p(f_i)}{\text{meas. } f_i} \right| = 0.$$

Hence it will be sufficient to show that $\lim p(f_i)/\text{meas. } f_i = A$.

Given f_i we can find a finite plurisegment f'_i, $f'_i \subset f_i$, such that

$$\nu(f'_i) \leq p(f_i) < \nu(f'_i) + \epsilon_i, \qquad (\epsilon_i > 0), \tag{1.2}$$

* Proved also by my colleague H. E. Bray, some time ago, but not published. It applies of course in n dimensions.

and if we let $f_i'' = f_i - f'_i$, f_i'' will be a finite plurisegment, and

$$\nu(f'_i) + \nu(f_i'') = p(f_i) - n(f_i) = \nu(f_i).$$

Hence, by subtraction,

$$(1.3) \qquad -\nu(f_i'') \leqq n(f_i) < -\nu(f_i'') + \epsilon_i.$$

We take ϵ_i small, so that $\lim\ (i = \infty)\ \epsilon_i/\omega_i = 0$.

We note first that the family f_i'' can contain no subsequence which constitutes a regular family with respect to the intervals ω_i, that is, allow the selection of a subsequence f_j'' such that

$$\lim_{j=\infty} \frac{\text{meas. } f_j''}{\omega_j} = k'' > 0.$$

For in this case,

$$\frac{n(f_j)}{\text{meas. } f_j} \leqq -\frac{\nu(f_j'')}{\text{meas. } f_j''} \frac{\text{meas. } f_j''}{\omega_j} \frac{\omega_j}{\text{meas. } f_j} + \frac{\epsilon_j}{\omega_j} \frac{\omega_j}{\text{meas. } f_j}.$$

As j tends to ∞, $\nu(f_j'')/\text{meas. } f_j''$ approaches A, and the first term in the right-hand member of the above inequality eventually becomes < 0, and hence

$$\leqq -\frac{\nu(f_j'')}{\text{meas. } f_j''} \cdot \frac{\text{meas } f_j''}{\omega_j}.$$

Consequently, as j tends to infinity,

$$0 \leqq \overline{\lim} \frac{n(f_j)}{\text{meas. } f_j} \leqq -Ak'' + 0$$

which is a contradiction.

It follows that $\lim\ (i = \infty)$ meas. $f_i''/\omega_i = 0$, and therefore that $\underline{\lim}$ meas. $f'_i/\omega_i = k$. Consequently

$$(1.4) \qquad \lim_{i=\infty} \frac{\text{meas. } f_i''}{\text{meas. } f_i} = 0, \qquad \lim_{i=\infty} \frac{\text{meas. } f'_i}{\text{meas. } f_i} = 1.$$

From (1.2) we have

$$\frac{\nu(f'_i)}{\text{meas. } f'_i} \frac{\text{meas. } f'_i}{\text{meas. } f_i} \leqq \frac{p(f_i)}{\text{meas. } f_i} < \frac{\nu(f'_i)}{\text{meas. } f'_i} \cdot \frac{\text{meas. } f'_i}{\text{meas. } f_i} + \frac{\epsilon_i}{\omega_i} \frac{\omega_i}{\text{meas. } f_i},$$

and from (1.4), letting i tend to infinity,

$$A \leqq \underline{\lim_{i=\infty}} \frac{p(f_i)}{\text{meas. } f_i} \leqq \overline{\lim} \frac{p(f_i)}{\text{meas. } f_i} \leqq A.$$

Hence $p(r)$ has the unique derivative A. And from

$$\frac{n(r_i)}{\text{meas. } r_i} = \frac{p(r_i)}{\text{meas. } r_i} - \frac{\nu(r_i)}{\text{meas. } r_i}$$

it follows that $n(r)$ has the unique derivative zero.

A similar proof holds when $A < 0$, but if $A = 0$ the method must be slightly different, since both f'_i and f_i'' may now contain subsequences which are regular families with respect to the intervals ω_i. If f_i'' (f'_i) contains no regular family of parameter $> k/2$, with respect to the intervals ω_i, the entire sequence f'_i (f_i'') is a regular family with parameter $\geqq k/2$ with respect to these intervals, and the proof proceeds as before. On the other hand, if both f'_i and f_i'' contain regular families of this sort, f'_j and f_k'' respectively, we show directly that

$$\lim_{j=\infty} \frac{p(f_j)}{\text{meas. } f_j} = 0, \qquad \lim_{k=\infty} \frac{n(f_k)}{\text{meas. } f_k} = 0$$

from which, since $A = 0$,

$$\lim_{j=\infty} \frac{n(f_j)}{\text{meas. } f_j} = 0, \qquad \lim_{k=\infty} \frac{p(f_k)}{\text{meas. } f_k} = 0.$$

But the entire sequence $\{i\}$ is exhausted by the sequence $\{j\}$ and the sequence $\{k\}$, since for every i, either meas. f'_i/meas. f_i or meas. f_i''/meas. $f_i \geq 1/2$. Hence for the entire sequence $\{i\}$

$$\lim_{j=\infty} \frac{p(f_i)}{\text{meas. } f_i} = \lim_{i=\infty} \frac{n(f_i)}{\text{meas. } f_i} = 0.$$

In order to prove that $t(r)$ has the unique derivative $|A|$ at $x = 0$, it is sufficient to remark that $t(r) = p(r) + n(r)$. Thus the proof of the lemma is complete.

In a Stieltjes integral, generalized or not, the function which appears in the integrand, before the differential sign, must be defined for every value of x in the interval to be considered. Accordingly we extend the definition of $\gamma(x)$ and write

$$\gamma(x)/x]_{x=0} = \lim_{\underline{x=0}} \gamma(x)/x, \tag{1.5}$$

admitting $+\infty$ as a possible value of this inferior limit.* With this extension, the following proposition is a consequence of Lemmas II and III.

LEMMA IV. *Let $g(x)$ be a function of bounded variation, $0 \leq x \leq a$, and $t(x)$ its total variation function. If $g(x)$ has a unique derivative at $x = 0$ (on regular families), namely $g'(0) = A$, then the generalized integrals*

$$\int_0^\delta [\gamma(x)/x]\, dg(x) = m_1(\delta), \quad \int_0^\delta [\gamma(x)/x]\, dt(x) = m_2(\delta), \quad 0 < \delta < a,$$

are convergent, and

* On account of the restriction imposed on $g(x)$ at $x = 0$, the truth of Lemma IV is independent of the definition of $\gamma(0)/0$, which may be arbitrary.

(1.6) $$\lim_{\delta=0} m_1(\delta) = 0, \qquad \lim_{\delta=0} m_2(\delta) = 0.$$

Since $t(x)$, by Lemma III, has a unique derivative at $x = 0$, it satisfies the conditions for $g(x)$, and it will be sufficient to prove the theorem for $g(x)$. We suppose first that $A = 0$, so that, choosing $g(0) = 0$,

$$g(x) = x\eta(x)$$

where $\eta(x)$ is of bounded variation in any closed interval (δ', δ) with $0 < \delta' < \delta$, and $|\eta(x)| < \eta_\delta$, $0 < x \leqq \delta$, with $\lim (\delta = 0)\ \eta_\delta = 0$. Without loss of generality we may assume $g(x)$ to be non-decreasing and $\eta(x)$ not negative, since if $g(x)$ has a zero derivative on regular families at $x = 0$, its positive and negative variation functions have also.

The generalized integral is given as the limit *

$$\lim_{n=\infty} \int_0^\delta \gamma_n(x)\, dg(x)$$

where $\gamma_n(x)$ is a sequence of functions, integrable with respect to $g(x)$, increasing with n, with $\lim_{n=\infty} \gamma_n(x) = \gamma(x)/x$, $0 \leq x \leq \delta$, and such that for each n, $\int_0^\delta \gamma_n(x)\, dg(x)$ exists.

We take $n > 1/\delta$, N as the lower bound of $\gamma(x)/x$ in the interval $0 \leq x \leq 1/n$, and write

$$\gamma_n(x) = N, \qquad 0 \leq x \leq 1/n$$
$$\gamma_n(x) = \gamma(x)/x, \qquad 1/n < x \leq \delta.$$

The function $\gamma_n(x)$ is bounded and measurable in the Borel sense, and is therefore integrable with respect to $g(x)$. For $x = 0$,

$$\lim_{n=\infty} \gamma_n(x) = \lim_{x=0} \gamma(x)/x = \gamma(x)/x]_{x=0},$$

and for $x > 0$, $\gamma_n(x) = \gamma(x)/x$ for $n > 1/x$; hence $\lim_{n=\infty} \gamma_n(x) = \gamma(x)/x$, $0 \leq x \leq \delta$. Moreover $\gamma_{n+k}(x) \geq \gamma_n(x)$ if $k > 0$. We have $N \leq n\gamma(1/n)$.

With these definitions we have

$$\int_0^\delta \gamma_n(x)\, dg(x) = N \int_0^{1/n} dg(x) + \int_{1/n}^\delta \gamma_n(x)\, dg(x)$$
$$= Ng(1/n) + \int_{1/n}^\delta \gamma_n(x)\, dg(x)$$

* P. J. Daniell, *loc. cit.* (I), p. 289.

$$\leq Ng(1/n) + \int_{1/n}^{\delta} [\gamma(x)/x]\, dg(x).$$

But

$$Ng(1/n) \leq n\gamma(1/n) \cdot (1/n)\eta(1/n) < \eta_\delta \gamma(\delta)$$

and by Lemma II

$$\int_{1/n}^{x} \gamma(x)/x\, dg(x) = \int_{1/n}^{\delta} \gamma(x)\, d\eta(x) + \int_{1/n}^{\delta} [\gamma(x)/x]\, \eta(x)\, dx$$

$$\leq \left| \int_{1/n}^{\delta} \gamma(x) d\eta(x) \right| + m(\delta)\eta_\delta$$

where $m(\delta) = \int_0^{\delta} \gamma(x)/x\, dx$. Moreover, since $\gamma(x)$ is monotonic, an application of Lemma II to $\int_{1/n}^{\delta} d\,[\gamma(x)\eta(x)]$ yields the identity

$$\int_{1/n}^{\delta} \gamma(x)\, d\eta(x) = \gamma(\delta)\eta(\delta) - \gamma(1/n)\eta(1/n) - \int_{1/n}^{\delta} \eta(x)\, d\gamma(x),$$

and therefore the inequality

$$\left| \int_{1/n}^{\delta} \gamma(x)\, d\eta(x) \right| < 2\gamma(\delta)\eta_\delta + \eta_\delta \gamma(\delta) = 3\gamma(\delta)\eta_\delta.$$

Hence

$$\int_0^{\delta} \gamma_n(x)\, dg(x) < \eta_\delta\, [m(\delta) + 4\gamma(\delta)].$$

But the integral is an increasing function of n. Accordingly the integral of the Lemma exists; moreover from the inequality just written it approaches zero with δ. This is what was to be proved.

Suppose now that $A \neq 0$, and write

$$g(x) = xA + [g(x) - xA].$$

We know, as the property (A) of the general integral, that $\int_0^{\delta} [\gamma(x)/x]\, dg(x)$ exists and has the value

$$\int_0^{\delta} [\gamma(x)/x]\, dg(x) = \int_0^{\delta} [\gamma(x)/x]\, d(xA) + \int_0^{\delta} [\gamma(x)/x]\, d\,[g(x) - xA],$$

provided that both integrals of the second member of this equation exist. The existence of the second integral has just been demonstrated; it approaches zero with δ. The first integral of the right-hand member evidently can be shown to exist by the same sort of proof, since either xA or $-xA$ is a non-decreasing function. The integral is seen to have the value $Am(\delta)$, and also approaches zero with δ.

In fact, we have (taking for simplicity $A > 0$),

$$\int_0^\delta \gamma_n(x)\,d(xA) = N \cdot A \cdot 1/n + \int_{1/n}^\delta \gamma_n(x) A\,dx$$
$$= A\gamma(1/n) + A\int_{1/n}^\delta \gamma(x)/x\,dx$$
$$= A\gamma(1/n) + A\,[m(\delta) - m(1/n)]$$

which has the limit $Am(\delta)$ as n tends to ∞.

An obvious corollary of the theorem which has just been demonstrated is the following:

COROLLARY 1. *Let $\phi(x)$ be summable in the Lebesgue sense, in the interval $(0, a)$, and let it be the derivative of its integral at $x = 0$. Then the integral*

$$\int_0^\delta \phi(x)\gamma(x)/x\,dx$$

is convergent, as a Lebesgue integral, and approaches zero with δ.

Since $\gamma(kx)$, where k is a given positive constant, is a function $\bar{\gamma}(x)$, with the properties described for $\gamma(x)$, we may note the following fact.

COROLLARY 2. *Lemma IV remains valid if $\gamma(x)$ is replaced by $\gamma(kx)$, k const., > 0.*

We denote the corresponding $m_2(\delta)$ by $m_3(\delta)$.

2. *Convergence of the integrals* (A), (B) *on the boundary.* At the position when $P = Q$ the quotients $\cos(QP, n_P)/QP$, $\cos(QP, n_Q)/QP$ are not defined. In order to fix these quantities, we may for the purpose of the theorems of this note assign them any values whatever. We choose more or less arbitrarily the value zero.

If C has a unique tangent at Q and the condition(γ) is satisfied at Q, there will be a neighborhood of Q on C for which the projection of an element Δs of arc, on the tangent line, will have a measure as near Δs as we please. We take coördinates x, y with the origin at Q, and the x-axis as the direction of the curve in the positive sense, at Q; and choose δ small enough so that if $|x| \leq \delta$, $\gamma(s) \leq 1/2$. Let C_δ be the connected portion of C containing Q for which $|x_P| \leq \delta$. For convenience in reference we recall the following facts,* of which the verification is immediate.

For $P = (x, y)$ in C_δ,

* Evans and Miles, *loc. cit.*, p. 495.

$$
\begin{aligned}
&\cos(n_Q, n_P) \geq 1/2, \qquad ds \leq 2dx, \qquad s \leq 2x \\
&|\, dy \,| \leq \gamma(s)\, ds \leq 2\gamma(2\,|\,x\,|\,)\,|\,dx\,|\,; \qquad |\,y\,| \leq 2\,|\,x\,|\,\gamma(2\,|\,x\,|\,) \\
(2.1)\quad &|\cos(QP, n_Q)\,| \leq |\,y\,|\,/\,|\,x\,| \leq 2\gamma(2\,|\,x\,|\,) \\
&|\cos(QP, n_P)\,| \leq |\cos(QP, n_Q)\,| + |\cos(QP, n_P) - \cos(QP, n_Q)\,| \\
&\qquad\qquad \leq 2\gamma(2\,|\,x\,|\,) + |\sphericalangle(n_Q, n_P)\,| \leq 3\gamma(2\,|\,x\,|\,).
\end{aligned}
$$

By means of these inequalities and Lemma IV we can prove at once the following proposition.

LEMMA V. *Let the condition* (γ) *be satisfied at* Q, *and let* $\nu(e)$ *have at* Q *a unique derivative* (*on regular families*), $\nu'(Q) = A$, *with respect to* s. *Then the integrals*

$$
(2.2)\quad \zeta = \int_C [\cos(QP, n_P)/QP]\, d\nu(e_P), \quad \zeta_1 = \int_C [\cos(QP, n_Q)/QP]\, d\nu(e_P)
$$

are convergent, and

$$
(2.3)\qquad \zeta = \lim_{\delta=0} \zeta'_\delta, \qquad \zeta_1 = \lim_{\delta=0} \zeta'_{1\delta}
$$

where ζ'_δ, $\zeta'_{1\delta}$ *are the integrals* ζ, ζ_1 *except that they are extended merely over the portion* $C - C_\delta$ of C.

In fact if we let ζ_δ, $\zeta_{1\delta}$ be the corresponding integrals extended over C_δ, their integrands are dominated respectively by the functions $3\gamma(2\,|\,x\,|\,)/|\,x\,|$ and $2\gamma(2\,|\,x\,|\,)/|\,x\,|$; and if we let $t(e)$ be the total variation function of $\nu(e)$, we have the result that a dominant integral for both ζ_δ and $\zeta_{1\delta}$ is the following

$$
\int_{C_\delta} [3\gamma(2x)/x]\, dt(e) = 3\int_0^\delta [\gamma(2x)/x]\, dt_1(x) = 3m_3(\delta)
$$

where $t_1(x) = [t(s_1) + t(s_2)]$, s_1 being the arc QP and s_2 the arc P_1Q such that $x_{P_1} = -x_P$, s_1 and s_2 being closed sets; $t_1(x)$ has a derivative at $x = 0$. But, since the integrands of ζ_δ and $\zeta_{1\delta}$ are both measurable in the Borel sense, and this dominant integral is convergent and approaches zero with δ, by Lemma IV, Corollary 2, the integrals ζ_δ and $\zeta_{1\delta}$ are both convergent and approach zero with δ.* Thus the theorem is proved.

The necessity of a smoothness condition at Q, of the nature of (γ), is made evident by the following example, in which C has a continuously turning tangent at Q, and $\nu(e)$ has a derivative (equal to zero) at Q, but where the integral ζ is not convergent.

---

* Daniell, *loc. cit.* (I), p. 290.

Take axes as before, and mark points $x = 1/2,\ 1/3, \cdots 1/k, \cdots$. At the point P_k $(x = 1/k)$, draw a segment, with a negative slope, such that its normal n_k satisfies the relation $\cos(x, n_k) = 1/(\log k)^{\frac{1}{2}}$. At the mid-point of the interval (P_{k+1}, P_k) draw a segment with a slope $1/k^2$. The broken line formed of these successive segments, with alternately positive and negative slopes, forms a portion of C abutting on Q. It is rectifiable, and

$$\lim (P = Q) \mid \sphericalangle (n_Q, n_P) \mid = 0,$$

provided that the vertices are smoothed, but the curve does not satisfy a (γ) condition at Q.

We form the function $\nu(e)$ by putting a mass a_k at each point P_k, and choose a_k so that $\nu(e)$ has a zero derivative, with respect to arc length, at Q. In fact, we take $a_k = [k(k+1)(\log k)^{\frac{1}{2}}]^{-1}$.

We have

$$\nu(s)/s \leq (x/s)(1/x) \sum_{k=r}^{\infty} a_k$$

where P_{r+1} is the P_k of lowest index contained in s. Hence

$$\begin{aligned}
\lim_{s=0} [\nu(s)/s] &\leq 1 \cdot \lim_{r=\infty} (r+1) \sum_{k=r}^{\infty} [k(k+1)(\log k)^{\frac{1}{2}}]^{-1} \\
&\leq \lim_{r=\infty} (r+1) \sum_{k=r}^{\infty} [1/k - 1/(k+1)]\,[1/(\log k)^{\frac{1}{2}}] \\
&\leq \lim_{r=\infty} \frac{r+1}{(\log r)^{\frac{1}{2}}} \left\{ \left(\frac{1}{r} - \frac{1}{r+1}\right) + \left(\frac{1}{r+1} - \frac{1}{r+2}\right) + \cdots \right\} \\
&\leq \lim\ [1/(\log r)^{\frac{1}{2}}] = 0.
\end{aligned}$$

Since $\nu(e)$ is a function of positive type it follows that the derivative on regular families is also zero.*

For the integral ζ, however, we have $\zeta = \zeta_\delta + \zeta'_\delta$

$$\zeta_\delta = \int_Q^{Q_\delta} \frac{\cos(QP, n_P)}{QP}\ d\nu(e_P) = \sum_{k=2}^{\infty} \frac{k}{(\log k)^{\frac{1}{2}}}\ a_k$$

$$= \sum_{k=2}^{\infty} \frac{1}{(k+1)\log k},$$

which fails to be convergent.

In the example just given, the point charges may be replaced by uniform density distributions over intervals which become suitably short as x tends to

* De la Vallée Poussin, *Intégrales de Lebesgue, Fonctions d'ensemble, Classes de Baire*, Paris (1916), p. 60.

zero. In the following example the curve is still smoother and the mass distribution is uniform, but with a positive density N at $x = 0$. This time it is the integral ζ_1 which fails to be convergent.

Let $\lambda(x)$ be a continuous non-decreasing function of x, with $\lambda(0) = 0$, and such that $\int_0^\delta \lambda(x)/x\,dx$ fails to be convergent. Consider the curve C, symmetric about the y-axis, of which a portion near Q may be represented, for positive x, in the form $y = \int_0^x \lambda(x)\,dx$. Take $N > 0$ and let $\nu(e)$ be the function of sets N meas. $e = \nu(e)$, associated with the point function $g(P) = Ns$; it has the constant density N. We have, if the integral ζ_1 exists,

$$\zeta_{1\delta} = 2N\int_0^\delta [y/(x^2+y^2)]\,[1+\lambda(x)^2]^{\frac{1}{2}}\,dx > 2N\int_0^\delta [y/(x^2+y^2)]\,dx$$

$$> N\int_0^\delta (y/x^2)\,dx, \text{ for } \delta \text{ small enough so that } x > y.$$

But by an integration by parts

$$\int_{\delta'}^\delta yx^{-2}\,dx = [-(1/x)\int_0^x \lambda(x)\,dx]_{\delta'}^\delta + \int_{\delta'}^\delta x^{-1}\lambda(x)\,dx,$$

and

$$\int_0^\delta yx^{-2}\,dx = -(1/\delta)\int_0^\delta \lambda(x)\,dx + \lim_{\delta=0}\int_{\delta'}^\delta x^{-1}\lambda(x)\,dx.$$

Hence the integral ζ_1 is not convergent.

The Fredholm method of treating the Dirichlet and Neumann problems is thus seen to be limited necessarily to curves which satisfy, at least almost everywhere, a condition of the type of (γ). The remark applies whether the boundary values are continuous or discontinuous. When the condition (γ) is not satisfied, in the former case many methods are still applicable; in the latter, the use of the Green's function leads to a general class of functions in which such problems are uniquely solvable,* but the connection with potentials of distributions of matter is lost, and the direct generalization to three dimensions is no longer feasible, on account of the lack of regularity of the Green's function.† The condition (γ) seems therefore to be a natural setting for the Fredholm method of treating the discontinuous boundary value problems in three dimensions.

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* G. C. Evans, *The Logarithmic Potential Discontinuous Dirichlet and Neumann Problems*, New York (1927).

† J. J. Gergen, "Mapping of a General Type of Three Dimensional Region on a Sphere," *American Journal of Mathematics*, Vol. 52 (1930), pp. 197-224.

3. *Limiting values and the proof of Theorem I.* The determination of limiting values *in the wide sense* depends upon the following simple result. Let Q be the origin of coördinates, and $M = (x, y)$ a point such that $|y|/|x| \geq \tan q$ where q is a given constant, $0 < q < \pi/2$. Let P_1 be a point movable on the x-axis, of x coördinate x_1, and denote MP_1 by r_1. Then the maximum value of the ratio $QP_1/MP_1 = |x_1|/r_1$ is $\csc q$. In other words, this ratio is bounded as M approaches Q in the wide sense:

$$|x_1|/r_1 \leq \csc q \quad \text{if} \quad |y|/|x| \geq \tan q. \tag{3.1}$$

Consider now the portion C_δ of C about Q, where the condition (γ) is assumed to hold. With axes as in § 2, let r denote MP, as before, and let P_1 be the projection of P on the x-axis. For P in C_δ, we have

$$\begin{aligned} |r - r_1|\, r_1^{-1} \leq |y_P|\, r_1^{-1} &< 2|x_1|\,\gamma(2|x_1|) r_1^{-1} \\ &< 2\gamma(2|x_1|)\csc q \leq 1/2 \end{aligned}$$

if δ is small enough. Consequently, $|x_1|\, r^{-1} = |x_1|\, r_1^{-1}\,[1 + (r - r_1)/r_1]^{-1} \leq 2|x_1|\, r_1^{-1}$ and

$$|x_1|/r < 2 \csc q \tag{3.2}$$

for all M such that $|y|/|x| \geq \tan q$, provided that δ is taken small enough, depending on q but not on M.

We note also the following inequalities, in which the same notation is conserved, and ϕ_1 denotes the angle (r_1, n_Q). We have

$$\begin{aligned} |r^{-1}\sin\phi' - r_1^{-1}\sin\phi_1| &= |x_1 - x|\;|r^{-2} - r_1^{-2}| \\ &= r_1^{-2} r^{-2}\,|x_1 - x|\;(r + r_1)\,|r - r_1| \\ &\leq 2|y_P|\, r^{-1} r_1^{-1} < \frac{2|x_1|\,\gamma(2|x_1|)x_1}{r_1 x_1 r} \\ &< 4\csc^2 q\,\gamma(2|x_1|)/x_1. \end{aligned}$$

Similarly

$$\begin{aligned} |r^{-1}\cos\phi' - r_1^{-1}\cos\phi_1| &\leq \left|\frac{y_P - y}{r^2} + \frac{y}{r_1^2}\right| \\ &\leq \left|\frac{y_P}{r^2}\right| + \frac{|y|\;|r_1^2 - r^2|}{r^2 r_1^2} \\ &\leqq 3|y|_P \bar{r}^{-2}, \end{aligned}$$

where $\bar{r}$ is the smaller of the two values r, r_1. Hence

$$|r^{-1}\cos\phi' - r_1^{-1}\cos\phi_1| \leq 6\left|\frac{x_1 x_1\,\gamma(2|x_1|)}{\bar{r}\,\bar{r}\,|x_1|}\right| < 24\csc^2 q\,\gamma(2|x_1|)/|x_1|.$$

But

$$r^{-1} \mid \sin\phi - \sin\phi' \mid \leq 2r^{-1} \left| \cos\frac{\phi+\phi'}{2} \sin\frac{\phi-\phi'}{2} \right| \leq r^{-1} \mid \phi - \phi' \mid$$

$$< 2\gamma(2 \mid x_1 \mid) \csc q / \mid x_1 \mid$$

and

$$r^{-1} \mid \cos\phi - \cos\phi' \mid < 2\gamma(2 \mid x_1 \mid) \csc q / \mid x_1 \mid .$$

Consequently *there is a number* $p > 0$, *such that for* P *in* C_δ, *we have, for* δ *small enough, depending on* q *but not on* M,

$$\begin{aligned} &\mid r^{-1}\sin\phi' - r_1^{-1}\sin\phi_1 \mid , \quad \mid r^{-1}\cos\phi' - r_1^{-1}\cos\phi_1 \mid , \\ &\mid r^{-1}\sin\phi - r_1^{-1}\sin\phi_1 \mid , \quad \mid r^{-1}\cos\phi - r_1^{-1}\cos\phi_1 \mid , \\ &\qquad\qquad \text{all} < p\,\gamma(2 \mid x_1 \mid)/x_1. \end{aligned} \tag{3.3}$$

If we denote any of these quantities by $f(x_1)$, we have

$$\mid \int_{C_\delta} f(x_1)\, d\nu(e_P) \mid < p m_3(\delta), \tag{3.4}$$

by Lemma IV, Corollary 2.

On the basis of these inequalities consider the integral $u_2(M)$ in Theorem I (E_2), and denote by $u_{2\delta}$, $u'_{2\delta}$ the corresponding integrals over the portions C_δ and $C - C_\delta$ respectively of C. We write

$$\bar{u}_{2\delta}(M) = \int_{C_\delta} \frac{\sin\phi_1}{r_1}\, d\nu(e) = \int_{-\delta}^{\delta} \frac{\sin\phi_1}{r_1} d\nu(x_1)$$

with $\nu(x_1) = \nu(s_1)$, where s_1 is the portion of C_δ for which $x \leq x_1$.

We have, with M' the image of M, as in Theorem I, from (3.4),

$$\begin{gathered} \mid u_{2\delta}(M) - \bar{u}_{2\delta}(M) \mid < p m_3(\delta) \\ \mid u_{2\delta}(M') - \bar{u}_{2\delta}(M') \mid < p m_3(\delta) \\ \bar{u}_{2\delta}(M) - \bar{u}_{2\delta}(M') = 0 \end{gathered}$$

so that, given ϵ, we have, independently of the position of M such that $\mid y/x \mid \geq \tan q$,

$$\mid u_{2\delta}(M) - u_{2\delta}(M') \mid < \epsilon/2$$

if δ is taken small enough so that $p < \epsilon/(4m_3(\delta))$.

We may now take M near enough to Q so that

$$\mid u'_{2\delta}(M) - u'_{2\delta}(M') \mid < \epsilon/2,$$

since these integrals are continuous as M passes through Q. Consequently, by taking M near enough to Q,

$$\mid u_2(M) - u_2(M') \mid < \epsilon.$$

But this is what was to be proved for (E_2) of Theorem I.

The proof of Theorem I (E_3), for $u_3(M)$, is identical with that for $u_2(M)$ just given. The examination of $u(M)$ and $u_1(M)$, however, requires a little more detail. Consider first the quantity

$$\bar{u}_{1\delta}(M) = \int_{C_\delta} \frac{\cos\phi_1}{r_1}\, d\nu(e) = \int_{C_\delta} \frac{\cos\phi_1}{r_1}\, d\nu(x_1)$$

where the function of bounded variation $\nu(x_1)$ has A for the value of its derivative at $x_1 = 0$. We have

$$\begin{aligned}\bar{u}_{1\delta}(M) &= \int_{C_\delta} \frac{\cos\phi_1}{r_1} A\, dx_1 + \int_{C_\delta} \frac{\cos\phi_1}{r_1} d\{\nu(x_1) - Ax_1\} \\ &= -A\theta(M,\delta) + \xi,\end{aligned}$$

$\theta(M,\delta)$ being the angle subtended by the directed segment $-\delta \leq x_1 \leq \delta$ at M, and

$$|\xi| \leq \int_{-\delta}^{\delta} r_1^{-1} |\cos\phi_1|\, dt(x_1)$$

where $t(x_1)$ is a function of bounded variation with a zero derivative at $x_1 = 0$;

$$t(x_1) = x_1\eta(x_1); \quad 0 \leq \eta(x_1) < \eta_\delta \quad \text{if} \quad -\delta \leq x_1 \leq \delta; \quad \lim_{\delta=0} \eta_\delta = 0.$$

If $|x| \geq \delta$, $r_1^{-1}\cos\phi_1$ is monotonic in $-\delta \leq x \leq \delta$. Otherwise it is monotonic in each of the sub-intervals $(-\delta, x)$ and (x, δ). We write

$$|\xi| \leq \left|\int_{-\delta}^{x} r_1^{-1}\cos\phi_1\, dt(x_1)\right| + \left|\int_{x}^{\delta} r_1^{-1}\cos\phi_1\, dt(x_1)\right|$$

$$\int_x^\delta \frac{\cos\phi_1}{r_1}\, dt(x_1) = t(x_1)\frac{\cos\phi_1}{r_1}\Big]_x^\delta - \int_x^\delta x_1\eta(x_1)\, d\,\frac{\cos\phi_1}{r_1}.$$

Now

$$\left| t(x_1)\frac{\cos\phi_1}{r_1}\Big]_x^\delta \right| = \left| \frac{x_1}{r_1}\eta(x_1)\cos\phi_1\Big]_x^\delta \right| \leq 2\eta_\delta \csc q$$

$$\left|\int_x^\delta x_1\eta(x_1)\, d\left(\frac{\cos\phi_1}{r_1}\right)\right| < \eta_\delta \left| \int_x^\delta |x_1|\, d\,\frac{\cos\phi_1}{r_1}\right| = \eta_\delta \left| |x_1|\,\frac{\cos\phi_1}{r_1}\Big]_x^\delta - \int_x^\delta \frac{\cos\phi_1}{r_1}\, d|x_1| \right|$$

$$< 2\eta_\delta \csc q + \eta_\delta \pi/2.$$

Consequently, whether $|x| \gtrless \delta$,

$$|\xi| \leq \eta_\delta \{8 \csc q + \pi\}$$

and

(3.5) $\quad |\bar{u}_{1\delta}(M) + A\theta(M,\delta)| \leq p_1\eta_\delta, \qquad p_1$ const., depending on q.

But from (3.3) and Lemma V,

$$\begin{aligned} | u_\delta(M) - \bar{u}_{1\delta}(M) | &< pm_3(\delta) \\ \zeta_\delta(M) &< 3m_3(\delta). \end{aligned}$$

Hence, given ϵ, we can take δ, small enough, depending on q but not on M, provided that $| y/x | \geq \tan q$, so that

$$\begin{aligned} | u_\delta(M) + A\theta(M,\delta) | &\leq \epsilon/3 \\ | \zeta_\delta | &\leq \epsilon/3. \end{aligned}$$

And since $u'_\delta(M)$ is continuous as M passes through Q, we can take M near enough to Q so that

$$| u(M) - u_\delta(M) - (\zeta - \zeta_\delta) | \leq \epsilon/3.$$

By taking M near enough to Q, therefore, with $| y/x | \geq \tan q$, we have

$$| u(M) + A\theta(M,\delta) - \zeta | \leq \epsilon.$$

Similarly

$$| u_1(M) + A\theta(M,\delta) - \zeta | \leq \epsilon.$$

Since, however, no matter what the value of δ,

$$\lim_{M=Q} \theta(M,\delta) = \pm\pi,$$

the above inequalities imply the relations (E), (E_1) of Theorem I. This completes the proof of the theorem.

4. *Potentials in the neighborhood of a vertex.* We may speak of a Q of C as a *spiral* point, if one of the four quantities $\bar{\theta}_+(Q)$, $\bar{\theta}_-(Q)$, $\underline{\theta}_+(Q)$, $\underline{\theta}_-(Q)$, defined in the Introduction, as we approach Q from one direction or the other on C, is infinite. We say that Q, on the other hand, is a vertex of C, if at Q the forward and backward directions of the curve exist in the ordinary sense, and if these directions are not the same and do not differ by π; in the former case Q is a *cusp* point. We denote the directions of the forward and backward tangents at Q by t_+ and t_- respectively. From these the directions of the interior "normals" n_{Q+} and n_{Q-} are established. The positive angle α from the forward tangent t_+ to the backward tangent t_- is called the angle at the vertex.

We may say that the curve satisfies a condition (γ_α) at Q if the angle at Q is α and if, with the function $\gamma(s)$ as already defined,

$$(4.1) \qquad | \sphericalangle (n_{Q\pm}, n_P) | < \gamma(s), \quad P \text{ on } C \text{ in } \pm \text{ sense from } Q.$$

The function $\nu(e)$ has a unique forward (backward) derivative on regular families at Q if it has no point value at Q, and if the limit is unique for families of sets e_i on the forward (backward) branch of the curve from Q, regular with respect to the smallest intervals ω_i which contain Q and e_i.

With these definitions we may state the proposition about the convergence of the integral $u(M)$, when $M = Q$, a vertex.

LEMMA V. *Let the condition* (γ_α) *be satisfied at* Q, *and* $\nu(e)$ *have unique forward and backward derivatives,* A_+ *and* A_- *respectively, (on regular families) at* Q. *Then the integral* ζ *converges, and*

$$\zeta = \lim_{\delta=0} \zeta'_\delta.$$

By C_δ we may understand the portion of C in the neighborhood of Q whose projections on t_+ and t_- have measures δ_+ and δ_- respectively, both $< \delta$.

Similarly (3. 3) applies, where the angle q is the lower bound of the value of the smallest absolute angle which QM makes with the directions t_+ or t_-, and n_{Q+} or n_{Q-} is used according as P is forward or backward from Q.

We are thus led to a useful theorem. Let M in T_+ or in T_- approach Q in such a way that QM approaches a limiting direction which makes an angle β with t_+, the angle β being positive, $0 < \beta < \alpha$, if M is in T_+ and negative, $\alpha - 2\pi < \beta < 0$, if M is in T_-.

THEOREM II. *If* C *satisfies* (γ_α) *at* Q *and* $\nu(e)$ *has unique forward and backward derivatives,* A_+ *and* A_- *respectively (on regular families) at* Q, *then*

$$\lim_{M=Q\pm} u(M) = (\beta \mp \pi)A_+ + (\alpha - \beta - \pi)A_- + \zeta. \tag{4. 2}$$

A special case of this theorem is the following corollary:

COROLLARY. *If* $A_+ = A_- = A$

$$\lim_{M=Q+} u(M) = (\alpha - 2\pi)A + \zeta, \qquad \lim_{M=Q-} u(M) = \alpha A + \zeta, \tag{4. 3}$$

and the limits exist in the wide sense.

5. *Extensions and applications.* The methods used in this paper permit the extension of the results without serious difficulty to three dimensions. Thus in both two and three dimensions they constitute a natural medium for the setting of the Fredholm method. In fact, in the case where the (γ) condition is satisfied uniformly for all points Q of C and $\nu(e)$ is absolutely continuous—which includes as a special case the continuous boundary value conditions—the Fredholm theory may be established more systematically and just as simply as for boundaries of class C''.

There is also an immediate application to the Cauchy integral extended over the boundary C. If this integral is written in terms of a boundary function $f + ig$, summable (L) on C, for points M in T_+,

$$w(z) = \frac{1}{2\pi i} \int_C \frac{f(t) + ig(t)}{t - z}\, dt,$$

a necessary and sufficient condition that $w(z)$ take on the values $f(t) + ig(t)$ in the wide sense, almost everywhere, or in fact, on a set of positive measure, on C, is that $w(z)$ vanish identically for z outside C. This result depends on the uniqueness theorem of Riesz and Lusin and Priwaloff,* and is established immediately by writing $w(z)$ in terms of integrals of the form $u(M)$ and $u_2(M)$, $M = z$, and applying Theorem I above. In obtaining this result there is thus involved an extension of the method of Fichtenholz † for the corresponding theorem in the case of the circle. Nevertheless, the latter method uses the inverse point and the special formulae related to the circle, and so is not directly applicable. Likewise, whereas in the case of the circle the class of functions involved is obtained by explicit formulae, here, results must be obtained from general considerations of the class of functions given by integrals like $u(M)$, $u_2(M) - u_2(M')$. If the (γ) condition is uniform, this class may be determined.

These results will be demonstrated in later notes.

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* N. Luzin and J. Priwaloff, *Annales scientifiques de l'École Normale Supérieure*, Vol. 41 (1925), p. 159.

† G. Fichtenholz, *Fundamenta Mathematicae*, Vol. 13 (1929), pp. 1-33 (see p. 23).

GENERALIZATIONS TO HIGHER DIMENSIONS OF THE CAUCHY INTEGRAL FORMULA.

By D. G. Fulton and G. Y. Rainich.

In the first part of the paper formulas are presented which are immediate generalizations to higher dimensions of the Cauchy integral formula giving the value of an analytic function in a point surrounded by a contour or closed hypersurface in terms of its value on the contour or hypersurface. Then in connection with Volterra's theory of conjugate functions, about which a brief outline is given, formulas are written which are extensions of a much more general nature of the Cauchy integral formula.

Immediate Generalizations to Higher Dimensions.

1. The generalizations may be better approached from an historical point of view. We may start with Newton's Law of Gravitation which says that the components of the gravitational field produced by a single material point are proportional to

$$x/r^3, \quad y/r^3, \quad z/r^3.$$

It can be readily verified that such a field (and also a sum of any number of such fields) satisfies the differential equations

$$X_1 + Y_2 + Z_3 = 0, \quad X_2 = Y_1, \quad X_3 = Z_1, \quad Y_3 = Z_2 \tag{1.1}$$

where X_1 means $\partial X/\partial x$, X_2 means $\partial X/\partial y$, etc. For this reason we call the set of equations (1.1) the Newtonian differential equations. It is clear that by setting $Z = 0$ we obtain a special case in which X and Y are functions of x and y alone, and satisfy the equations

$$X_1 + Y_2 = 0, \quad X_2 = Y_1$$

which may be interpreted as the Cauchy-Riemann equations for the function

$$W = Y + iX.$$

Thus we see that the Newtonian equations are a generalization of the Cauchy-Riemann equations. Later we shall take up still more general cases; at present we want to derive a formula which bears the same relationship to the Newtonian equations as the Cauchy integral formula bears to the Cauchy-Riemann equations.

2. We begin by rewriting the Cauchy integral formula

$$f(t) = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{z-t}$$

in a form which readily yields itself to generalization. Making $t = 0$ we have

$$f(0) = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{z}\ .$$

We write $f(z) = W = Y + iX$, $z = x + iy$, $dz = dx + idy$ and $-i(dx + idy) = (\alpha + \beta i)\,ds$, where α and β are the direction cosines of the normal to the contour, and the formula becomes

$$2\pi X_0 = \int\left(X\,\frac{\alpha x + \beta y}{r^2} + Y\,\frac{\beta x - \alpha y}{r^2}\right) ds.$$

$$2\pi Y_0 = \int\left(X\,\frac{\alpha x + \beta y}{r^2} + Y\,\frac{\alpha x + \beta y}{r^2}\right) ds.$$

Once expressed in this manner it is easy to prove the formula. We shall give the proof here because the proof in higher cases will be modelled on this one.

The proof consists of the following two steps.

a). The integrals vanish on a contour not surrounding the origin because by the divergence theorem

$$\int [X(\alpha x + \beta y)/r^2 + Y(\beta x - \alpha y)/r^2]\,ds$$
$$= \int \{(\partial/\partial x)[(Xx - Yy)/r^2] + (\partial/\partial y)[(Xy + Yx)/r^2]\}\,dA$$

and the integrand of the last integral equals

$$(x/r^2)(\partial X/\partial x + \partial Y/\partial y) + (y/r^2)(\partial X/\partial y - \partial Y/\partial x)$$
$$+ X[\partial(x/r^2)/\partial x + \partial(y/r^2)/\partial y] + Y[\partial(x/r^2)/\partial y - \partial(y/r^2)/\partial x]$$

which vanishes since each parenthesis is zero.

b). It follows that the value of the integral is the same for all contours around the origin; we may integrate, for instance, around a circle of radius r. We have in this case $x/r = \alpha$, $y/r = \beta$, which makes the first integral become $\int X(ds/r)$ and we know that it is independent of r, whence it is easily seen that its value is $2\pi X_0$.

3. We can now generalize the formula to the Newtonian case. By analogy we write:

$$4\pi X_0 = \int\left(X\,\frac{\alpha x + \beta y + \gamma z}{r^3} + Y\,\frac{\beta x - \alpha y}{r^3} + Z\,\frac{\gamma x - \alpha z}{r^3}\right) d\omega$$

$$4\pi Y_0 = \int \left(X \frac{\alpha y - \beta x}{r^3} + Y \frac{\alpha x + \beta y + \gamma z}{r^3} + Z \frac{\gamma y - \beta z}{r^3} \right) d\omega$$

$$4\pi Z_0 = \int \left(X \frac{\alpha z - \gamma x}{r^3} + Y \frac{\beta z - \gamma y}{r^3} + Z \frac{\alpha x + \beta y + \gamma z}{r^3} \right) d\omega.$$

The proof of this formula is exactly similar to that in two dimensions.

The integral formula in two dimensions may be considered as a limit of this one in three dimensions.

4. The three integrals above may be expressed very easily in one integral. Let

$$W = iX + jY + kZ, \qquad n = i\alpha + j\beta + k\gamma$$

and

$$M = i\,[\partial(1/r)/\partial x] + j\,[\partial(1/r)/\partial y] + k\,[\partial(1/r)/\partial z].$$

Then we have

$$4\pi W_0 = \int W \cdot n \cdot M \cdot d\omega.$$

The product under the integral sign is to be understood as the product of quaternions.*

5. The generalization to higher dimensions is obvious. For four dimensions, for example, we consider four functions X, Y, Z, T and set up as analogous to the Newtonian equations the system

$$X_1 + Y_2 + Z_3 + T_4 = 0$$

$$X_2 = Y_1, \quad X_3 = Z_1, \quad X_4 = T_1, \quad Y_3 = Z_2, \quad Y_4 = T_2, \quad Z_4 = T_3.$$

In this case we get in the same fashion,

$$2\pi^2 X_0 = \int \left(X \frac{\alpha x + \beta y + rz + \delta t}{r^4} + Y \frac{\beta x - \alpha y}{r^4} + Z \frac{\gamma x - \alpha y}{r^4} + T \frac{\delta x - \alpha t}{r^4} \right) d\tau \dagger \text{ etc.}$$

The proof again does not present any difficulty.

6. For n dimensions we consider n functions F_i, and we have the system of equations ‡

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* A. C. Dixon, *Quarterly Journal*, Vol. 35 (1904), pp. 283-296. Also D. Iwanenko and K. Nikolski, *Zeitschrift für Physik*, Vol. 63 (1930), p. 129.

† For the measure of a hypersphere in n dimensions see Somerville, "An Introduction to the Geometry of n Dimensions."

‡ Here and in what follows a Greek letter used as an index means summation with respect to that index.

$$\partial F_\sigma/\partial x_\sigma = 0, \qquad \partial F_i/\partial x_j = \partial F_j/\partial x_i$$
$$(\sigma = \sum 1, 2, 3 \cdots n;\ i, j = 1, 2, 3 \cdots n).$$

The corresponding integral formula is

$$\frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}(F_i)_0 = \int\left(F_i \frac{\alpha_\sigma x_\sigma}{r^n} + F_\sigma \frac{\alpha_\sigma x_i - \alpha_i x_\sigma}{r^n}\right)d\tau.$$

This may be considered as the immediate generalizations of the Cauchy integral formula but it is possible to apply the same principles to a still more general case.

More General Extensions to Higher Dimensions.

7. In 1889, Volterra * founded the theory of conjugate functions of which all the preceding cases are special cases but which contain many others. We adopt the integral point of view in order to show in what this generalization consists. The systems of functions F_i which we considered so far give zero integrals on contours and closed hypersurfaces, that is, for spaces of n dimensions on closed manifolds of 1 and $n-1$ dimensions. Volterra's generalization consists in considering systems which give zero integrals on closed manifolds of r and $n-r$ dimensions.

Without going into details (which the reader may look up in a paper by F. D. Murnaghan in the *Physical Review* (1921), pp. 73-87 and in his book *Vector Analysis and the Theory of Relativity*, pp. 72-77), we mention the following points:

The functions in this case will be $F_{i_1 i_2 \ldots i_r}$, which change the sign if two indices are interchanged, and consequently vanish when two indices are identical. These functions give rise to another system of functions $R_{i_1 i_2 \ldots i_{n-r}}$ by means of the formulas

$$F_{i_1 i_2 \ldots i_r} = R_{i_{r+1} i_{r+2} \ldots i_n},$$

where the n subscripts form an even permutation of the numbers $1, 2, 3, \cdots n$.

The system lends itself to integration over an r-dimensional surface and the integral will be

$$\int F_{\rho_1 \rho_2 \ldots \rho_r}\, \alpha_{\rho_1 \rho_2 \ldots \rho_r} \tag{7.1}$$

where

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* V. Volterra, "Sulle funzioni conjugate," *Rendiconti dei Lincei* (4), Vol. 5 (1889), pp. 599-611. For developments analogous to those presented here see Volterra's paper in *Rendiconti di Palermo*, Vol. 3 (1889), pp. 260-272.

$$\alpha_{i_1 i_2 \ldots i_r} = \begin{vmatrix} dx_{i_1}, & dx_{i_2}, & \cdots, & dx_{i_r} \\ \delta_1 x_{i_1}, & \delta_2 x_{i_2}, & \cdots, & \delta_1 x_{i_r} \\ \delta_2 x_{i_1}, & \delta_2 x_{i_2}, & \cdots, & \delta_2 x_{i_r} \\ \cdot & \cdot & \cdot & \cdot \\ \delta_{r-1} x_{i_1}, & \delta_{r-1} x_{i_2}, & \cdots, & \delta_{r-1} x_{i_r} \end{vmatrix}.$$

On an $n - r$ manifold the integral will be of the form

$$\int R_{\rho_1 \rho_2 \ldots \rho_{n-r}} \alpha_{\rho_1 \rho_2 \ldots \rho_{n-r}} \tag{7.2}$$

where $\alpha_{i_1 i_2 \ldots i_{n-r}}$ is similar to $\alpha_{i_1 i_2 \ldots i_r}$. Using the divergence theorem

$$\int_V \frac{\partial F_{i_1 i_2 \ldots i_{r-1} i\rho}}{\partial x_\rho} \, d\tau = \int_S F_{i_1 i_2 \ldots i_{r-1} i\rho} \alpha_{i_1 i_2 \ldots i_{r-1} i\rho}$$

and an analogous relation for the R's, we find the following system of differential equations to be equivalent to the vanishing of the integrals (7.1) and (7.2):

$$\frac{\partial F_{i_1 i_2 \ldots i_{r-1} i\rho}}{\partial x_\rho} = 0, \tag{7.3}$$

$$\frac{\partial R_{i_1 i_2 \ldots i_{n-r-1} i\rho}}{\partial x_\rho} = 0. \tag{7.4}$$

Making use of the relationship between the F's and the R's we may write (7.4) as

$$(-1)^\rho \frac{\partial F_{i_1 i_2 \ldots i' \rho \ldots i_{r+1}}}{\partial x_\rho} = 0 \tag{7.5}$$

where a primed index means that all the indices appear with the exception of the one primed.

8. It is clear that these equations are generalizations of the Cauchy-Riemann and the Newtonian equations. Besides they contain many other cases of which the simplest is that corresponding to $n = 4$, $r = 2$. The set of differential equations giving the vanishing of surface integrals is

$$\partial F_{ia}/\partial x_a = 0, \qquad \partial R_{ia}/\partial x_a = 0.$$

These equations differ only in sign before certain terms from the fundamental equations of electro-dynamics in free space in Maxwell's theory. This difference in sign is essential for our purposes since it would substitute for integrals over a closed surface integrals over a surface extending to infinity—compare for the case of Maxwell's equations with the article by Murnaghan referred to above.

We derived these equations as conditions for the vanishing of *surface*

integrals, but they also may be shown (by applying the reasoning used at the end of section 7) to be conditions for the vanishing of certain *hypersurface* integrals, namely, the eight integrals

$$\int F_{i\tau}\alpha_\tau d\omega \quad \text{and} \quad \int R_{i\tau}\alpha_\tau d\omega \qquad (i=1,2,3,4).$$

We may expect that for these equations the integral formula will be analogous to the Cauchy formula; in fact, we find

$$2\pi^2(F_{ij})_0 = \int\left(F_{ij}\frac{\alpha_\tau x_\tau}{r^4} + F_{\tau j}\frac{\alpha_\tau x_i - \alpha_i x_\tau}{r^4} + F_{i\tau}\frac{\alpha_\tau\alpha_j - \alpha_j x_\tau}{r^4}\right) d\omega$$

where α_i $(i=1,2,3,4)$ are the direction cosines of the normal to the hypersurface.

9. In the case where n is arbitrary and $r=3$ we may write the integral in a form similar to the ones for $r=2$. We get

$$\frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}(F_{ijk})_0 = \int\Big(F_{ijk}\frac{\alpha_\tau x_\tau}{r^n} + F_{\tau jk}\frac{\alpha_\tau x_i - \alpha_i x_\tau}{r^n} + F_{i\tau k}\frac{\alpha_\tau x_j - \alpha_j x_\tau}{r^n} + F_{ij\tau}\frac{\alpha_\tau x_k - \alpha_k x_\tau}{r^n}\Big) d\omega.$$

The proof of this formula consists of two steps as in the two-dimensional case.

(9.1) The value of the integral is independent of the hypersurface surrounding the origin. To prove this we show that

$$\sum_{l=1}^{n}\frac{\partial(\text{coef. of } \alpha_l)}{\partial x_l} = 0.$$

Coefficient of α_l,

$l=i$: $F_{\tau jk}(x_\tau/r^n)$
$l=j$: $F_{i\tau k}(x_\tau/r^n)$
$l=k$: $F_{ij\tau}(x_\tau/r^n)$
$l\neq ijk$: $F_{ijk}(x_l/r^n) - F_{ljk}(x_i/r^n) - F_{ilk}(x_j/r^n) - F_{ijl}(x_k/r^n)$

(a) $$\sum_{l=1}^{n}\frac{\partial(\text{coef. of } \alpha_l)}{\partial x_l} = \sum_{l=1}^{n}\left(\frac{x_l}{r^n}\frac{\partial F_{ijk}}{\partial x_l} - \frac{x_i}{r^n}\frac{\partial F_{ljk}}{\partial x_l} - \frac{x_j}{r^n}\frac{\partial F_{ilk}}{\partial x_l} - \frac{x_k}{r^n}\frac{\partial F_{ijl}}{\partial x_l}\right) \qquad (l\neq i,j,k).$$

(b)$^+$ $$(x_\tau/r^n)(\partial F_{\tau jk}/\partial x_i + \partial F_{i\tau k}/\partial x_j + \partial F_{ij\tau}/\partial x_k)$$

(c)$^+$ $$\sum_{l=1}^{n}(F_{ijk}/r^n) + 3F_{ijk}/r^n \qquad (l\neq i,j,k).$$

(d)$^+$ $$\sum_{l=1}^{n}[\partial(1/r^n)/\partial x_l]\{F_{ijk}x_l - x_iF_{ljk} - x_jF_{ilk} - x_kF_{ijl}\} \qquad (l\neq i,j,k).$$

(e)$^+$ $[\partial(1/r^n)/\partial x_i]\, x_\tau F_{\tau jk} + [\partial(1/r^n)/\partial x_j]\, x_\tau F_{i\tau k} + [\partial(1/r^n)/\partial x_k]\, x_\tau F_{ij\tau}$.

Lines (a) and (b) combined vanish as a result of the differential equations

$$\partial F_{i_1 i_2 i_\rho}/\partial x_\rho = 0,$$

$$(-1)^\rho \frac{\partial F_{i_1 i_2 \ldots i'_\rho \ldots i_4}}{\partial x_\rho} = 0.$$

In (d) and (e) combined ($l \neq i, j, k$ in the next five formulas)

$$\sum_{l=1}^{n} [\partial(1/r^n)/\partial x_l]\, x_l F_{ijk} = -(n/r^{n+2}) F_{ijk} \sum_{l=1}^{n} x_l^2 \ ;$$

$$-x_i \sum_{l=1}^{n} \{[\partial(1/r^n)/\partial x_l]\, F_{ljk}\} + [\partial(1/r^n)/\partial x_i]\, x_\tau F_{\tau jk} = -n(x_i^2/r^{n+2}) F_{ijk};$$

$$-x_j \sum_{l=1}^{n} \{[\partial(1/r^n)/\partial x_l]\, F_{ilk}\} + [\partial(1/r^n)/\partial x_j]\, x_\tau F_{i\tau k} = -n(x_j^2/r^{n+2}) F_{ijk};$$

$$-x_k \sum_{l=1}^{n} \{[\partial(1/r^n)/\partial x_l]\, F_{ijl}\} + [\partial(1/r^n)/\partial x_k]\, x_\tau F_{ij\tau} = -n(x_k^2/r^{n+2}) F_{ijk}.$$

Hence (c), (d) and (e) together give

$$\sum_{l=1}^{n} (F_{ijk}/r^n) + 3F_{ijk}/r^n - (nF_{ijk}/r^{n+2}) \Big(\sum_{l=1}^{n} x_l^2 + x_i^2 + x_j^2 + x_k^2 \Big) = 0.$$

Therefore

$$\sum_{l=1}^{n} \frac{\partial(\text{coef. of } \alpha_l)}{\partial x_l} = 0.$$

(9.2) Since it is true that the value of the integral is the same on all hypersurfaces we may integrate around a hypersphere in n dimensions. Here $x_i/r = \alpha_i$, and the integrand reduces to F_{ijk}/r^{n-1}. Since the integral is independent of r we easily obtain as its value

$$\frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)} (F_{ijk})_0.$$

The same reasoning would apply to the most general case; namely, when both n and r are arbitrary, $r < n$. The integral formula in this case is

$$\frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)} (F_{i_1 i_2 \ldots i_r})_0 = \int \Big(F_{i_1 i_2 \ldots i_r} \frac{\alpha_\tau x_\tau}{r^n} + F_{\tau i_2 i_3 \ldots i_r} \frac{\alpha_\tau x_{i_1} - \alpha_{i_1} x_\tau}{r^n}$$

$$+ F_{i_1 \tau i_3 \ldots i_r} \frac{\alpha_\tau x_{i_2} - \alpha_{i_2} x_\tau}{r^n} + \cdots + F_{i_1 i_2 \ldots i_{r-1} \tau} \frac{\alpha_\tau x_{i_r} - \alpha_{i_r} x_\tau}{r^n} \Big) d\omega.$$

This is the general formula we wanted to obtain. All the preceding formulas are special cases of it.

FOURIER DEVELOPMENTS FOR CERTAIN PSEUDO-PERIODIC FUNCTIONS IN TWO VARIABLES.

By MIGUEL A. BASOCO.

1. *Introduction.* Jacobi,* in his investigations on the dynamics of a rotating rigid body, was led to the discovery of certain pseudo-periodic functions which, using Hermite's † nomenclature, belong to the class of doubly periodic functions of the second kind. Expressed in terms of the Jacobi theta functions, they have the form

$$\phi_{\alpha\beta\gamma}(x, y) \equiv \vartheta'_1 \frac{\vartheta_\alpha(x+y)}{\vartheta_\beta(x)\vartheta_\gamma(y)}, \tag{1}$$

where x and y are independent complex variables, $\vartheta'_1 = \vartheta_0\vartheta_2\vartheta_3$, and (α, β, γ) are restricted to certain sixteen triads ‡ out of the possible sixty-four which can be selected from the numbers 0, 1, 2, 3. Hermite and Kronecker § obtained the Fourier series developments of these functions; recently, these have been recalculated and corrected by Bell,¶ who has shown that they play

* Jacobi, *Werke*, Bd. 2, pp. 291-351.

† Hermite, "Sur quelques applications des fonctions elliptiques," *Comptes Rendus*, Vol. 85 (1877),. . ., Vol. 94 (1882); *Oeuvres*, Vol. 3, p. 267; "Sur une application de la théorie des fonctions doublement périodiques de seconde espèce," *Annales de l'École Normale Supérieure* (3), Vol. 2 (1885), p. 303; *Oeuvres*, Vol. 4, pp. 190 and 199-200.

‡ The series for the functions corresponding to the remaining forty-eight triads have not been calculated; the method due to Teixeira [see footnote †, p. 243] indicates why only sixteen appear in a natural manner; the remaining can not be obtained by his method. However, by multiplying each of the known $\phi_{\alpha\beta\gamma}(x, y)$ by a suitable Jacobian elliptic function, it is possible to obtain expansions for the forty-eight missing functions, *altered*, however, by a multiplicative factor of the second degree in the theta constants ϑ_α, $\alpha = 0, 2, 3$. From a different point of view, these functions have recently been studied by D. A. F. Robinson in a University of Chicago doctoral dissertation (1930). His results are to be published in a forthcoming number of the *Proceedings of the Royal Society of Canada, Toronto.* The series obtained by these methods have a more complex structure than those corresponding to the $\phi_{\alpha\beta\gamma}(x, y)$ as defined in equation (1).

§ Kronecker, *Monatsberichte der Berliner Akademie* (1881), pp. 1165 to 1172; *Werke*, Bd. IV, pp. 309 to 318. See also Tannery-Molk, *Fonctions Elliptiques*, Vol. 3, pp. 120-131.

¶ E. T. Bell, "Arithmetical Paraphrases, II," *Transactions of the American Mathematical Society*, Vol. 22, No. 2 (1921), p. 207; *Colloquium Publications*, Vol. 7, p. 88. See also, *Messenger of Mathematics*, Vol. 49 (Sept., 1919), p. 83.

an all-important rôle in certain investigations in the theory of numbers, as for example, in establishing some of the very general formulae involving numerical functions which were published by Liouville * without proof or indication as to their source.

A systematic study of the doubly periodic functions of the second kind, as a class, has been made by F. Gomes Teixeira.† He has given a general theory for obtaining the trigonometric expansions of these functions and has indicated, by an example, how it may be applied to the functions $\phi_{\alpha\beta\gamma}(x, y)$. He takes, as a starting point, the function of x, $\vartheta_\alpha(x+y)/\vartheta_\beta(x)$, regarding y temporarily as a parameter; his analysis then leads in a direct manner to the series for the functions (1).

In the present paper we obtain the arithmetized Fourier developments of certain functions $\Theta_{\alpha\beta\gamma}(x, y)$, which are related to the $\phi_{\alpha\beta\gamma}(x, y)$ by the identity

$$\Theta_{\alpha\beta\gamma}(x, y) \equiv \frac{\vartheta'_1}{\vartheta_\beta(x)} \cdot \phi_{\alpha\beta\gamma}(x, y) = \vartheta'^2_1 \frac{\vartheta_\alpha(x+y)}{\vartheta_\beta^2(x)\vartheta_\gamma(y)} . \tag{2}$$

In as much as the arithmetized series for the indicated factors in (2) are known,‡ this identity may be used to obtain arithmetical paraphrases of the general Liouville type. These, as well as other such formulae derivable from the $\Theta_{\alpha\beta\gamma}(x, y)$ will involve *incomplete* numerical functions in *two variables.* So far as the writer is aware no expansions involving incomplete numerical functions in *more than one variable* have appeared in the literature. The importance of such functions may be judged from a recent paper by Uspensky,§ who by purely arithmetic methods has established certain formulae of the paraphrase type which have reference to incomplete functions in three variables.

As suggested by Teixeira's example, mentioned above, we begin with $\vartheta_\alpha(x+y)/\vartheta_\beta^2(x)$ regarding it as a function of x only; since, as will be evident from what follows, these are *not* doubly periodic of the second kind,

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* Liouville, "Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres," Eighteen memoirs in *Journal de Mathématiques* (1858-1865).

† Teixeira, "Sur le developpement des fonctions doublement périodiques de seconde espèce en serie trigonometrique," *Crelle's Journal für Mathematik,* Bd. 125 (1901), pp. 301-318.

‡ E. T. Bell, "Theta Expansions Useful in Arithmetic," *Messenger of Mathematics,* Vol. 54 (1924).

§ J. V. Uspensky, *Bulletin of the American Mathematical Society,* Vol. 36 (October, 1930), p. 743; *American Journal of Mathematics,* Vol. 50 (1928), No. 1. See also an article by A. Oppenheim in *Quarterly Journal of Mathematics,* Oxford series, Vol. 2, no. 7, (Sept., 1931).

Teixeira's method is not applicable and hence our analysis will run along somewhat different lines. The method * which we shall use is not limited in its application to the particular functions here treated, but may also be applied to functions of the form $\vartheta_\alpha(x+y)/[\vartheta_\beta(x)]^r$, where r is an integer greater than unity. For $r>2$, however, the results are fairly complex.†

As in the case of the $\phi_{\alpha\beta\gamma}(x,y)$, our method leads to sixteen functions $\Theta_{\alpha\beta\gamma}(x,y)$; from these, sixteen more may be obtained by permuting the variables x and y. The remaining ninety-six may be obtained from these thirty-two by multiplying them by suitable elliptic functions. In this connection, see note (‡), page 242.

2. *Group I.* The functions $\Theta_{\alpha\beta\gamma}(x,y)$ appear in four groups of four each, such that those within a given group may be derived from any one in the same group by replacing the variables (x,y) in turn by $(x+\pi/2,y)$, $(x,y+\pi/2)$ and $(x+\pi/2,\ y+\pi/2)$. To obtain the expansions for the functions in the first group, we may begin with

$$f(x)=\frac{\vartheta_1(x+y)}{\vartheta_1^{\,2}(x)}, \tag{3}$$

which, as follows from the properties of the theta functions,‡ satisfies the relations

$$\begin{cases} f(x+\pi)=-f(x);\ f(x+2\pi)=f(x) \\ f(x+n\pi\tau)=(-1)^n q^{n^2}e^{2ni(x-y)}f(x), \qquad [i=(-1)^{1/2}]. \end{cases} \tag{4}$$

where, $q=e^{i\pi\tau}$, $0<\operatorname{amp}\tau<\pi$, so that $|q|<1$, and n is any positive or negative integer, including zero. The Laurent expansion of $f(x)$ relative to the pole $x=0$ is readily seen to be

$$A_2/x^2+A_1/x+P(x), \tag{5}$$

in which $A_2=\vartheta_1(y)/\vartheta'_1{}^2$ and $A_1=\vartheta'_1(y)/\vartheta'_1{}^2$.

Consider a parallelogram $ABCD$ in the x-complex plane, composed of $n+1$ double cells (of width 2π) above and n below the real axis. In this parallelogram, let the poles of the function in the r-th double cell have affixes $a_r=r\pi\tau$ and $b_r=\pi+r\pi\tau$. From (4) and (5) it follows that in the neighborhood of these poles, $f(x)$ is represented by the series

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* See Whittaker and Watson, *Modern Analysis*, Fourth Edition, p. 139.

† The method is theoretically applicable to functions of the form

$$[\vartheta_\alpha(x+y)]^s/[\vartheta_\beta(x)]^r,$$

provided $s<r$.

‡ We use the small theta notation of Jacobi, with his ϑ replaced by ϑ_0, see *Werke*, I, s. 501. In this notation the argument of the circular functions does not, as in some others, contain the factor π.

$$(6)\qquad \begin{cases} (-1)^r q^{r^2} e^{-2ir\nu}\left(\dfrac{A_2}{(x-a_r)^2}+\dfrac{A_1+2irA_2}{x-a_r}+\cdots\right), \\ -(-1)^r q^{r^2} e^{-2ir\nu}\left(\dfrac{A_2}{(x-b_r)^2}+\dfrac{A_1+2irA_2}{x-b_r}+\cdots\right), \end{cases}$$

respectively.

We now consider the auxiliary function

$$(7)\qquad \Phi(t)=\frac{f(t)}{\sin(t-x)}.$$

Within the parallelogram $ABCD$, this function has poles at $t=x$, $t=\pi+x$, $t=a_r$, $t=b_r$, $r=0, \pm 1, \pm 2, \cdots, \pm n$. (Whenever any of the poles lie on the boundary of the parallelogram, as is the case here, it is permissible to derange the mesh formed by the period cells so that all poles may lie within.) The residues relative to the first two poles are the same and equal to $f(x)$; from (6) and (7) it may be readily seen that the residues relative to a_r and b_r are, in both cases,

$$(-1)^r q^{r^2} e^{-2ir\nu}[(A_1+2irA_2)\operatorname{cosec}(a_r-x)-A_2\operatorname{cosec}(a_r-x)\operatorname{ctn}(a_r-x)].$$

On applying Cauchy's theorem to the function $\Phi(t)$, with $ABCD$ as contour of integration and then allowing n to become infinite we obtain equation (8) below. Clearly, the integral may be replaced by four other integrals, two of which will cancel due to the periodicity of $\Phi(t)$, while the other two will tend to zero as n tends to infinity, due to the presence of the factor q^{n^2}. Obviously, it is the existence of such a factor as this, which makes this method of expansion possible. We may, thus, write

$$(8)\qquad \begin{cases} \vartheta_1'^2\dfrac{\vartheta_1(x+y)}{\vartheta_1^2(x)\vartheta_1(y)}=\dfrac{\vartheta_1'(y)}{\vartheta_1(y)}\cdot\sum_{r=-\infty}^{\infty}(-1)^r q^{r^2}e^{-2ir\nu}\operatorname{cosec}(x-r\pi\tau) \\ \qquad\qquad +2i\sum_{r=-\infty}^{\infty}(-1)^r q^{r^2} r e^{-2ir\nu}\operatorname{cosec}(x-r\pi\tau) \\ \qquad\qquad +\sum_{r=-\infty}^{\infty}(-1)^r q^{r^2}e^{-2ir\nu}\operatorname{cosec}(x-r\pi\tau)\operatorname{ctn}(x-r\pi\tau). \end{cases}$$

This expansion is valid for all values of x and y with the exception of those for which the function is not defined. For arithmetical purposes, the expansion, as just obtained, is not useful; it is necessary to transform it into a Fourier series. This may be accomplished by replacing each sum in (8) by two others in which the indices of summation range from $r=1$ to $r=\infty$, pairing off corresponding terms and making use of the following developments:

$$\text{(9)} \qquad \operatorname{cosec}(x + r\pi\tau) = -2i \sum_{m=1}^{\infty} q^{mr} e^{mix}, \qquad (m = 1, 3, 5, 7, \cdots),$$

$$\text{(10)} \qquad \operatorname{cosec}(x - r\pi\tau) = 2i \sum_{m=1}^{\infty} q^{mr} e^{-mix},$$

$$\text{(11)} \qquad \operatorname{cosec}(x + r\pi\tau)\operatorname{ctn}(x + r\pi\tau) = -2 \sum_{m=1}^{\infty} m q^{mr} e^{mix},$$

$$\text{(12)} \qquad \operatorname{cosec}(x - r\pi\tau)\operatorname{ctn}(x - r\pi\tau) = -2 \sum_{m=1}^{\infty} m q^{mr} e^{-mix}.$$

On making the necessary reductions and applying the remarks made at the beginning of this section we obtain the expansions corresponding to the functions in the first group. They are valid for all values of y for which the functions are defined and for all values of x such that

$$-I(\pi\tau) < I(x) < I(\pi\tau),$$

i. e., such that their representative points in the x-plane lie within a strip whose boundaries are straight lines parallel to the axis of reals and passing through the points $x = \pi\tau$ and $x = -\pi\tau$.

With regard to notation we point out that the finite sums which appear as the coefficients of q^r, refer to all the integer divisors d, δ or t, τ of r subject to whatever restrictions may be indicated. We shall let n be an unrestricted integer greater than zero, m an odd integer > 0, α and β positive integers of the form $4k + 1$ and $4k + 3$, respectively; the divisor τ shall always be odd and positive.

Group I.

$$(2n = d\delta,\ d < (2n)^{\frac{1}{2}},\ \delta - d \equiv 1,\ \mod 2.)$$

$$(I_a): \begin{cases} \Theta_{111}(x, y) = \vartheta_1'^2 \dfrac{\vartheta_1(x+y)}{\vartheta_1^2(x)\vartheta_1(y)} = \dfrac{\cos x}{\sin^2 x} - 4 \sum q^{2n} \\ \qquad \left(\sum (-1)^d (d+\delta) \cos((\delta - d)x + 2dy) \right) \\ \quad + \dfrac{\vartheta_1'(y)}{\vartheta_1(y)} \left(\dfrac{1}{\sin x} + 4 \sum q^{2n} \left(\sum (-1)^d \sin((\delta - d)x + 2dy) \right) \right). \end{cases}$$

$$(I_b): \begin{cases} \Theta_{221}(x, y) = \vartheta_1'^2 \dfrac{\vartheta_2(x+y)}{\vartheta_2^2(x)\vartheta_1(y)} = -\dfrac{\sin x}{\cos^2 x} + 4 \sum q^{2n} \\ \qquad \left(\sum (-1)^{(d+\delta-1)/2} (d+\delta) \sin((\delta - d)x + 2dy) \right) \\ \quad + \dfrac{\vartheta_1'(y)}{\vartheta_1(y)} \left(\dfrac{1}{\cos x} + 4 \sum q^{2n} \left(\sum (-1)^{(d+\delta-1)/2} \cos((\delta - d)x + 2dy) \right) \right. \end{cases}$$

$$(I_c): \quad \begin{cases} \Theta_{212}(x,y) = \vartheta_1'^2 \dfrac{\vartheta_2(x+y)}{\vartheta_1^2(x)\vartheta_2(y)} = \dfrac{\cos x}{\sin^2 x} - 4\sum q^{2n} \\ \qquad \left(\sum(d+\delta)\cos((\delta-d)x+2dy)\right) \\ \quad + \dfrac{\vartheta_2'(y)}{\vartheta_2(y)}\left(\dfrac{1}{\sin x} + 4\sum q^{2n}\left(\sum \sin((\delta-d)x+2dy)\right)\right). \end{cases}$$

$$(I_d): \quad \begin{cases} \Theta_{122}(x,y) = \vartheta_1'^2 \dfrac{\vartheta_1(x+y)}{\vartheta_2^2(x)\vartheta_2(y)} = \dfrac{\sin x}{\cos^2 x} - 4\sum q^{2n} \\ \qquad \left(\sum(-1)^{(\delta-d-1)/2}(d+\delta)\sin((\delta-d)x+2dy)\right) \\ \quad - \dfrac{\vartheta_2'(y)}{\vartheta_2(y)}\left(\dfrac{1}{\cos x} + 4\sum q^{2n}\left(\sum(-1)^{(\delta-d-1)/2}\cos((\delta-d)x+2dy)\right)\right). \end{cases}$$

3. *Group II.* The details involved in the calculation of the expansions for the functions of this group are entirely similar to those in the preceding section and therefore will be omitted. We find that if we begin with $f(x) = \vartheta_2(x+y)/\vartheta_0^2(x)$ we are led to the series

$$(13) \quad \begin{cases} \vartheta_1'^2 \dfrac{\vartheta_2(x+y)}{\vartheta_0^2(x)\vartheta_3(y)} = -\dfrac{\vartheta_3'(y)}{\vartheta_3(y)} \sum_{r=-\infty}^{\infty} q^{(2r+1)^2/4} e^{-i(2r+1)y} \operatorname{cosec}(x-(2r+1)\pi\tau/2) \\ \quad - i\sum_{r=-\infty}^{\infty} q^{(2r+1)^2/4}(2r+1)e^{-i(2r+1)y}\operatorname{cosec}(x-(2r+1)\pi\tau/2) \\ \quad - \sum_{r=-\infty}^{\infty} q^{(2r+1)^2/4} e^{-i(2r+1)y}\operatorname{cosec}(x-(2r+1)\pi\tau/2)\operatorname{ctn}(x-(2r+1)\pi\tau/2). \end{cases}$$

With the aid of the series similar to (9)—(12), the preceding expansion may be reduced to a Fourier series. Thus we get the following group, in which,

$$-\tfrac{1}{2} I(\pi\tau) < I(x) < \tfrac{1}{2} I(\pi\tau).$$

Group II.

$$(\beta = t\tau,\ \beta \equiv 3 \bmod 4,\ \tau < \beta^{1/2}.)$$

$$(II_a): \quad \begin{cases} \Theta_{203}(x,y) = \vartheta_1'^2 \dfrac{\vartheta_2(x+y)}{\vartheta_0^2(x)\vartheta_3(y)} \\ \qquad = 2\sum q^{\beta/4}\left(\sum(t+\tau)\cos\left(\dfrac{t-\tau}{2}x + \tau y\right)\right. \\ \qquad\quad - 4\dfrac{\vartheta_3'(y)}{\vartheta_3(y)} \sum q^{\beta/4}\left(\sum \sin\dfrac{t-\tau}{2}x + \tau y\right)\Big). \end{cases}$$

$$(II_b): \begin{cases} \Theta_{133}(x,y) = \vartheta_1'^2 \dfrac{\vartheta_1(x+y)}{\vartheta_3^2(x)\vartheta_3(y)} \\ \quad = 2\sum q^{\beta/4}\left(\sum(-1)^{(t-\tau-2)/4}(t+\tau)\sin\left(\frac{t-\tau}{2}x+\tau y\right)\right) \\ \quad + 4\dfrac{\vartheta_3'(y)}{\vartheta_3(y)}\sum q^{\beta/4}\left(\sum(-1)^{(t-\tau-2)/4}\cos\left(\frac{t-\tau}{2}x+\tau y\right)\right). \end{cases}$$

$$(II_d): \begin{cases} \Theta_{100}(x,y) = \vartheta_1'^2 \dfrac{\vartheta_1(x+y)}{\vartheta_0^2(x)\vartheta_0(y)} \\ \quad = 2\sum q^{\beta/4}\left(\sum(-1)^{(\tau-1)/2}(t+\tau)\sin\left(\frac{t-\tau}{2}x+\tau y\right)\right) \\ \quad + 4\dfrac{\vartheta_0'(y)}{\vartheta_0(y)}\sum q^{\beta/4}\left(\sum(-1)^{(\tau-1)/2}\cos\left(\frac{t-\tau}{2}x+\tau y\right)\right). \end{cases}$$

$$(II_c): \begin{cases} \Theta_{230}(x,y) = \vartheta_1'^2 \dfrac{\vartheta_2(x+y)}{\vartheta_3^2(x)\vartheta_0(y)} \\ \quad = -2\sum q^{\beta/4}\left(\sum(-1)^{(t+\tau)/4}(t+\tau)\cos\left(\frac{t-\tau}{2}x+\tau y\right)\right) \\ \quad + 4\dfrac{\vartheta_0'(y)}{\vartheta_0(y)}\sum q^{\beta/4}\left(\sum(-1)^{(t+\tau)/4}\sin\left(\frac{t-\tau}{2}x+\tau y\right)\right). \end{cases}$$

4. *Group III.* The functions of the first two groups have the period 2π while those in the next two have the period π. Hence, the appropriate auxiliary function $\Phi(t)$ to be used in connection with these last groups is

$$\Phi(t) = \frac{f(t)}{\tan(t-x)}, \tag{14}$$

in place of (7). Also, the contour of integration is modified so that the width of the parallelogram $ABCD$ in § 2, is π, i. e., the width of a period cell. Proceeding as before, if we let $f(x)$ be $\vartheta_0(x+y)/\vartheta_0^2(x)$, we are led to the series

$$(15) \quad \begin{cases} \vartheta_1'^2 \dfrac{\vartheta_0(x+y)}{\vartheta_0^2(x)\vartheta_1(y)} \\ = i\dfrac{\vartheta_1'(y)}{\vartheta_1(y)}\sum_{r=-\infty}^{\infty}(-1)^{r+1}q^{(2r+1)^2/4}e^{-i(2r+1)y}\operatorname{ctn}(x-(2r+1)\pi\tau/2) \\ \quad + \sum_{r=-\infty}^{\infty}(-1)^r q^{(2r+1)^2/4}(2r+1)e^{-i(2r+1)y}\operatorname{ctn}(x-(2r+1)\pi\tau/2) \\ \quad + i\sum_{r=-\infty}^{\infty}(-1)^{r+1}q^{(2r+1)^2/4}e^{-i(2r+1)y}\operatorname{cosec}^2(x-(2r+1)\pi\tau/2). \end{cases}$$

The reduction of this expression to a Fourier series can be carried out with the aid of the following expansions:

$$\text{(16)} \qquad \operatorname{ctn}(x - r\pi\tau) = i\,[1 + 2\sum_{n=1}^{\infty} q^{2rn}\, e^{-2inx}],$$

$$\text{(17)} \qquad \operatorname{ctn}(x + r\pi\tau) = -i\,[1 + 2\sum_{n=1}^{\infty} q^{2rn}\, e^{2inx}],$$

$$\text{(18)} \qquad \operatorname{cosec}^2(x - r\pi\tau) = -4\sum_{n=1}^{\infty} nq^{2rn}\, e^{-2inx},$$

$$\text{(19)} \qquad \operatorname{cosec}^2(x + r\pi\tau) = -4\sum_{n=1}^{\infty} nq^{2rn}\, e^{2nix}.$$

In the following series the variable x is restricted so that

$$-\tfrac{1}{2} I(\pi\tau) < I(x) < \tfrac{1}{2} I(\pi\tau).$$

Group III.

($\alpha = t\tau$, $\alpha \equiv 1 \bmod 4$, $\tau < \alpha^{1/2}$; $m \equiv 1 \bmod 2$, $(-1/m) = (-1)^{(m-1)/2}$.)

$$(III_a): \begin{cases} \Theta_{001}(x,y) = \vartheta_1'^2 \dfrac{\vartheta_0(x+y)}{\vartheta_0^2(x)\vartheta_1(y)} = 2\sum(-1/m)\, mq^{m^2/4} \sin my \\ \quad + 2\sum q^{\alpha/4}\left(\sum(-1/\tau)(t+\tau)\sin\left(\dfrac{t-\tau}{2}x + \tau y\right)\right) \\ \quad + \dfrac{\vartheta_1'(y)}{\vartheta_1(y)}\left\{ 2\sum(-1/m)\, q^{m^2/4}\cos my \right. \\ \quad \left. + 4\sum q^{\alpha/4}\left(\sum(-1/\tau)\cos\left(\dfrac{t-\tau}{2}x + \tau y\right)\right)\right\}. \end{cases}$$

$$(III_b): \begin{cases} \Theta_{331}(x,y) = \vartheta_1'^2 \dfrac{\vartheta_3(x+y)}{\vartheta_3^2(x)\vartheta_1(y)} = 2\sum(-1/m)\, mq^{m^2/4} \sin my \\ \quad + 2\sum q^{\alpha/4}\left(\sum(-1)^{(t+\tau-2)/4}(t+\tau)\sin\left(\dfrac{t-\tau}{2}x + \tau y\right)\right) \\ \quad + \dfrac{\vartheta_1'(y)}{\vartheta_1(y)}\left\{ 2\sum(-1/m)\, q^{m^2/4}\cos my \right. \\ \quad \left. + 4\sum q^{\alpha/4}\left(\sum(-1)^{(t+\tau-2)/4}\cos\left(\dfrac{t-\tau}{2}x + \tau y\right)\right)\right\}. \end{cases}$$

$$(III_c): \begin{cases} \Theta_{302}(x,y) = \vartheta_1'^2 \dfrac{\vartheta_3(x+y)}{\vartheta_0^2(x)\vartheta_2(y)} = 2\sum q^{m^2/4} m\cos my \\ \quad + 2\sum q^{\alpha/4}\left(\sum(t+\tau)\cos\left(\dfrac{t-\tau}{2}x + \tau y\right)\right) \\ \quad - \dfrac{\vartheta_2'(y)}{\vartheta_2(y)}\left\{ 2\sum q^{m^2/4}\sin my \right. \\ \quad \left. + 4\sum q^{\alpha/4}\left(\sum\sin\left(\dfrac{t-\tau}{2}x + \tau y\right)\right)\right\}. \end{cases}$$

$$(III_d): \begin{cases} \Theta_{032}(x,y) = \vartheta_1^2 \dfrac{\vartheta_0(x+y)}{\vartheta_3^2(x)\vartheta_2(y)} = 2\sum q^{m^2/4} m \cos my \\ \qquad + 2\sum q^{a/4}\left(\sum (-1)^{(t-\tau)/4}(t+\tau)\cos\left(\dfrac{t-\tau}{2}x+\tau y\right)\right) \\ \qquad -\dfrac{\vartheta_2'(y)}{\vartheta_2(y)}\left\{ 2\sum q^{m^2/4}\sin my \right. \\ \qquad \left. + 4\sum q^{a/4}\left(\sum (-1)^{(t-\tau)/4}\sin\left(\dfrac{t-\tau}{2}x+\tau y\right)\right)\right\}. \end{cases}$$

5. *Group IV.* If we let $f(x)$ be $\vartheta_0(x+y)/\vartheta_1^2(x)$ and proceed as in the preceding section we are led to the expansion

$$(20) \qquad \begin{cases} \vartheta_1^2 \dfrac{\vartheta_0(x+y)}{\vartheta_1^2(x)\vartheta_0(y)} = 2i\sum_{r=-\infty}^{\infty} (-1)^r r q^{r^2} e^{-2iry} \operatorname{ctn}(x - r\pi\tau) \\ \qquad + \sum_{r=-\infty}^{\infty} (-1)^r q^{r^2} e^{-2iry} \operatorname{cosec}^2(x - r\pi\tau) \\ \qquad + \dfrac{\vartheta_0'(y)}{\vartheta_0(y)} \sum_{r=-\infty}^{\infty} (-1)^r q^{r^2} e^{-2iry} \operatorname{ctn}(x - r\pi\tau). \end{cases}$$

Using (16) to (19) in connection with (20) we obtain the expansions in Group IV. Here the variable x is restricted by the condition

$$-I(\pi\tau) < I(x) < I(\pi\tau).$$

Group IV.

$$(n = d\delta,\ \delta - d \equiv 0 \bmod 2,\ d < n^{1/2}.)$$

$$(IV_a): \begin{cases} \Theta_{010}(x,y) = \vartheta_1^2 \dfrac{\vartheta_0(x+y)}{\vartheta_1^2(x)\vartheta_0(y)} = \dfrac{1}{\sin^2 x} - 4\sum_{n=1}^{\infty} (-1)^n n q^{n^2} \cos 2ny \\ \qquad - 4\sum q^n\left(\sum (-1)^d (d+\delta)\cos((\delta-d)x + 2dy)\right) \\ \qquad + \dfrac{\vartheta_0'(y)}{\vartheta_0(y)}\left\{ \operatorname{ctn} x + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}\sin 2ny \right. \\ \qquad \left. + 4\sum q^n\left(\sum (-1)^d \sin((\delta-d)x + 2dy)\right)\right\}. \end{cases}$$

$$(IV_b): \quad \left\{ \begin{aligned} \Theta_{320}(x,y) = \vartheta'^2_1 \frac{\vartheta_3(x+y)}{\vartheta_2^2(x)\vartheta_0(y)} &= \frac{1}{\cos^2 x} - 4\sum (-1)^n n q^{n^2} \cos 2ny \\ &- 4\sum q^n \left(\sum (-1)^{(d+\delta)/2} (d+\delta) \cos\left((\delta - d)x + 2dy\right) \right) \\ &- \frac{\vartheta'_0(y)}{\vartheta_0(y)} \left\{ \tan x - 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \sin 2ny \right. \\ &\left. - 4\sum q^n \left(\sum (-1)^{(d+\delta)/2} \sin\left((\delta - d)x + 2dy\right) \right) \right\}. \end{aligned} \right.$$

$$(IV_c): \quad \left\{ \begin{aligned} \Theta_{313}(x,y) = \vartheta'^2_1 \frac{\vartheta_3(x+y)}{\vartheta_1^2(x)\vartheta_3(y)} &= \frac{1}{\sin^2 x} - 4\sum_{n=1}^{\infty} n q^{n^2} \cos 2ny \\ &- 4\sum q^n \left(\sum (d+\delta) \cos\left((\delta - d)x + 2dy\right) \right) \\ &+ \frac{\vartheta'_3(y)}{\vartheta_3(y)} \left\{ \operatorname{ctn} x + 2\sum_{n=1}^{\infty} q^{n^2} \sin 2ny \right. \\ &\left. + 4\sum q^n \left(\sum \sin\left((\delta - d)x + 2dy\right) \right) \right\}. \end{aligned} \right.$$

$$(IV_d): \quad \left\{ \begin{aligned} \Theta_{023}(x,y) = \vartheta'^2_1 \frac{\vartheta_0(x+y)}{\vartheta_2^2(x)\vartheta_3(y)} &= \frac{1}{\cos^2 x} - 4\sum_{n=1}^{\infty} n q^{n^2} \cos 2ny \\ &- 4\sum q^n \left(\sum (-1)^{(\delta-d)/2} (d+\delta) \cos\left((\delta - d)x + 2dy\right) \right) \\ &- \frac{\vartheta'_3(y)}{\vartheta_3(y)} \left\{ \tan x - 2\sum q^{n^2} \sin 2ny \right. \\ &\left. - 4\sum q^n \left(\sum (-1)^{(\delta-d)/2} \sin\left((\delta - d)x + 2dy\right) \right) \right\}. \end{aligned} \right.$$

6. In the preceding expansions, some of the terms are multiplied by expressions of the form $\vartheta'_a(y)/\vartheta_a(y)$; the arithmetized series for these are well known * and we list them here for the sake of completeness.

$$\frac{\vartheta'_0(y)}{\vartheta_0(y)} = 4\sum q^n \left(\sum \sin 2ty \right), \qquad (n = t\tau), \tag{21}$$

$$\frac{\vartheta'_1(y)}{\vartheta_1(y)} = \operatorname{ctn} y + 4\sum q^{2n} \left(\sum \sin 2dy \right), \qquad (n = d\delta), \tag{22}$$

* See, for example Bell's "Theta Expansions Useful in Arithmetic," *loc. cit.*, p. 169. Also, certain reduction formulae in his paper in the *Giornale di Matematiche*, Vol. 61 (1923), pp. 213-228, will be found useful.

$$(23) \qquad \frac{\vartheta'_2(y)}{\vartheta_2(y)} = -\tan y + 4\sum q^{2n}\left(\sum (-1)^d \sin 2dy\right), \qquad (n = d\delta),$$

$$(24) \qquad \frac{\vartheta'_3(y)}{\vartheta_3(y)} = 4\sum (-1)^n q^n \left(\sum \sin 2ty\right), \qquad (n = t\tau).$$

Expansions (21) and (24) are valid for $-\frac{1}{2} I(\pi\tau) < I(y) < \frac{1}{2} I(\pi\tau)$, while (22) and (23) are valid for $-I(\pi\tau) < I(y) < I(\pi\tau)$.

When these are introduced into our expansions the coefficients of the general power of q, will then be a finite sum ranging over the solutions of certain *quaternary quadratic forms.*

7. In conclusion we may point out that our formulae will yield expansions for certain doubly periodic functions of the *third kind.* Thus, if we put $x = y$ in, say, (II_c) and apply the transformation of the second order, we obtain an expansion for $\vartheta'_1[\vartheta_1(x)\vartheta_2(x)\vartheta_3(x)/\vartheta_0^2(x)]$. If we let $x = -y$, we obtain series for functions of the form $\vartheta'_1{}^2\vartheta_2/\vartheta_3^2(x)\vartheta_0(x)$. Again, if we put $y = 0$, wherever it is permissible, as in (II_c), we get series for functions of the type $\vartheta'_1\vartheta_2\vartheta_3[\vartheta_1(x)/\vartheta_0^2(x)]$. With regard to the last two cases just mentioned, it should, perhaps, be stated that the method of the present paper, when applied to functions involving a single variable, does not differ in essence from the method given by Appell's theory * of the decomposition of doubly periodic functions of the third kind with more poles than zeros. For, as is shown by the writer elsewhere,† it is possible, by the method here indicated, to derive Appell's fundamental function and his formula of decomposition for this type of function.

Finally, the writer wishes to acknowledge his indebtedness to Professor E. T. Bell for his suggestion to the effect that the functions here treated might lead, when expanded into the series here given, to incomplete functions in more than one variable.

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* Appell, *Annales Scientifiques de l'École Normale Supérieure,* Vols. 1, 2, 3, 5, series 3, 1884-1888.

† *Acta Mathematica,* Vol. 57 (1931), pp. 95-100.

SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS BY MEANS OF HYPERVARIABLES.

By P. W. KETCHUM.

1. *Introduction.* The simplicity and elegance of the fact that the analytic functions of an ordinary complex variable constitute a general solution of Laplace's equation in two variables, have induced many attempts to obtain a similar general solution of Laplace's equation in three or more variables. B. Peirce * described a hypervariable which he thought would succeed. But Bechk-Widmanstetter † proved that no hypervariable of three units can be used. Whittaker ‡ by generalizing in another direction succeeded in obtaining a general solution by means of an integral. His work was extended by Bateman.§ An interesting general solution of Laplace's equation in four variables was obtained by Nicolesco.¶

The writer ‖ has obtained a hypervariable associated with an algebra containing an infinity of units such that the totality of analytic functions (*i. e.*, as in ordinary complex variables, functions expressible as power series or functions possessing unique derivatives) constitutes a general solution of Laplace's equation in three dimensions. The hypervariable is defined as follows:

$$w = ze_{-1} + xe_0 + ye_1$$

where e_{-1}, e_0, e_1 are three of an infinite set of units whose multiplication table may be defined by putting

$$e_0 = 1, \quad e_{-n} = i^n \sin n\alpha, \quad e_n = i^n \cos n\alpha,$$

[e. g., $e_{-4} \cdot e_3 = i^7 \sin 4\alpha \cdot \cos 3\alpha = \frac{1}{2} i^7 \sin 7\alpha - \frac{1}{2} i \sin \alpha = \frac{1}{2} e_{-7} - \frac{1}{2} e_{-1}$.]

It should be noticed that although w contains only three units, functions of w will in general contain more than three units. For example:

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* *Linear Associative Algebra*, p. 124.

† *Monatshefte für Mathematik und Physik*, Vol. 23 (1912), p. 257.

‡ *Mathematische Annalen*, Vol. 57 (1902), p. 333; Whittaker and Watson, *Modern Analysis*, p. 388.

§ *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 1 (1904), p. 451.

¶ *Comptes Rendus*, Vol. 185 (1927), p. 442. See also Hedrick and Ingold, *Transactions of the American Mathematical Society*, Vol. 27 (1925), p. 551.

‖ *American Journal of Mathematics*, Vol. 51 (1929), p. 179. The first equation in Section 3 should read $f(w) = \Sigma f_i(x, y, z) e_i$. Similar equations in the middle of p. 182 and at the bottom of p. 183 should read $f(w) = \Sigma f_i e_i$.

$$w^2 = \tfrac{1}{2}(2x^2 - y^2 - z^2)e_0 + 2xye_1 + 2xze_{-1} + \tfrac{1}{2}(y^2 - z^2)e_2 + yze_{-2}.$$

Functions other than polynomials will in general contain an infinity of units. In all cases the coefficients of the units satisfy Laplace's equation.

The formal relation between this solution and Whittaker's is somewhat like the relation between Fourier series and Fourier integrals, but the viewpoints are quite different. In the two dimensional case the real and imaginary parts of powers of a complex variable form the circular harmonics. In an analogous way, if

$$w^m = \sum_{j=-m}^{m} f_j(x, y, z)e_j,$$

the functions f_j are proportional to the solid spherical harmonics. For example, in the above expression for w^2 the coefficients of the e's are the solid spherical harmonics of order 2 except for constant factors. In fact this is perhaps the most natural and direct approach to the subject of spherical harmonics. Formally this result is already known and has been used as the basis of elementary expositions of the subject.*

In the present paper it is intended to make a further and more general study of the relations between linear partial differential equations with constant coefficients and hypervariables. In sections 3 and 4 it will be shown that to every such equation there is associated a linear associative and commutative algebra. Some properties of these algebras will be determined. In section 5 a hypervariable is found which gives a general solution of Airy's equation. Section 6 gives the hypervariables for Laplace's equation in n variables, including the wave equation as a special case. Section 7 contains an application to simultaneous partial differential equations.

2. *Form of the general solution of a differential equation.* Consider the homogeneous partial differential equation with constant coefficients:

$$P(\partial/\partial x_1, \partial/\partial x_2, \cdots \partial/\partial x_n) \cdot U = 0$$

where P is a homogeneous polynomial of degree r and with either real or ordinary complex coefficients. (Any equation with constant coefficients in n variables may be written as a homogeneous equation in $n+1$ variables.)†

By an appropriate shift of origin, U may be expanded in a power series in $x_1, x_2, \cdots x_n$,‡ uniformly and absolutely convergent in some region. In

* See the article on Spherical Harmonics in the *Encyclopedia Britannica*.

† Bateman, *ibid.*, p. 455.

‡ We consider only analytic solutions. For equations of the Laplace type all solutions must be analytic.

order to have an explicit expression for the number of terms of degree m in this series, we define

$$q_1(m) = 1, \quad q_n(m) = \sum_{s=0}^{m} q_{n-1}(s),$$

for $n = 2, 3, \cdots$; $m = 0, 1 \cdots$. Then there will be $q_n(m)$ terms of the power series of degree m. Let the homogeneous polynomial of degree m contained in the power series be denoted by h_m. Because the differential equation is homogeneous, h_m must itself be a solution. Operating on h_m by P gives a homogeneous polynomial g_m of degree $m - r$ for $m \geqq r$. g_m will contain $q_n(m - r)$ linearly independent terms. For h_m to be a solution, that is, for $P \cdot h_m$ to vanish identically, the coefficients of g_m must vanish, thus giving as many linear equations on the coefficients of h_m. It can be shown that these equations are linearly independent. (If $m < r$ there are no such equations, so we put $q_n(s) = 0$ if $s < 0$.)

Hence there will be $q_n(m) - q_n(m - r)$ (which we will denote by t) *linearly independent homogeneous polynomials of degree m which satisfy the equation* $P \cdot U = 0$, and *every homogeneous polynomial of degree m which is a solution of this equation may be expressed linearly in terms of such a fundamental set.** Furthermore, *any set of t linearly independent solutions,* u_{mi}, *which are homogeneous polynomials of degree m can be used as such a fundamental set, and*

$$\sum_{m=0}^{\infty} \sum_{i=1}^{t} c_{mi} u_{mi}$$

is a general solution in the neighborhood of the origin, the c's being arbitrary constants.

3. *Algebra associated with the general solution of a differential equation.* Consider the hypervariable

$$w = \sum_{j=1}^{n} x_j e_j$$

where the e's are numbers in a commutative and associative algebra, as yet undefined. Then for $m \geqq r$,

$$P(\partial/\partial x_1, \partial/\partial x_2, \cdots \partial/\partial x_n) \cdot w^m = [m!/(m - r)!]\, w^{m-r} P(e_1, e_2, \cdots e_n).$$

Hence w^m, *and all analytic functions* (expressible by power series) *of* w, *will satisfy the differential equation identically if and only if*

$$P(e_1, e_2, \cdots e_n) = 0 \tag{1}$$

This fundamental algebraic equation will be called the *auxiliary equation.*

* Bateman, *ibid.*

We are thus led to a study of the linear algebras formed by using $e_1, e_2, \cdots e_n$ as generators and having the following properties. It is understood that the algebra will be taken over the same field of numbers as the coefficients of the differential equation.

(a) Associative and commutative.

(b) $e_1, e_2, \cdots e_n$ satisfy (1).

(c) $e_1, e_2, \cdots e_n$ shall be the only generators, i. e., every number of the algebra is a polynomial or formal power series in $e_1, e_2, \cdots e_n$.

(d) There shall be no other relations between the generators independent of (a) and (b).

We now wish to find a set of *units* for the algebra. Let us first consider the special case of a first order differential equation. Then the auxiliary equation will be of the first degree, and a linear relation will hold between the numbers $e_1, \cdots e_n$. In other words these generators are not all independent and only $n-1$ of them can be taken as units of the algebra. Suppose we pick $e_1, \cdots e_{n-1}$ as units. (Any other set of $n-1$ linearly independent combinations of the numbers $e_1, \cdots e_n$ would evidently serve as well). Then all the products of the units $e_1, \cdots e_{n-1}$ disregarding order of multiplication, are linearly independent and can be taken as additional units of the algebra. For example, there will be $n(n-1)/2$ units of degree 2 in the generators, e_1^2, e_1e_2, etc. We thus get an infinite set of units, and at the same time the multiplication table is defined. It is seen that the units fall into groups according to their degree in the generators. Multiplication of a unit in a group corresponding to degree p in the generators by one of degree s gives a unit of degree $p+s$ in the generators.

For the general case of a differential equation of any order the situation is more complicated. The generators will themselves be linearly independent but some of their products will not be. The quantities $e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_n^{\alpha_n}$, where $\sum \alpha_i = m$, are all linearly independent if $m < r$. Hence these quantities may all be taken as independent units, there being $q_n(m)$ of them for each $m < r$. For $m \geqq r$, (b) gives $q_n(m-r)$ relations for each m between these quantities. Hence in any case we can choose for each m, t *quantities of the form* $e_1^{\alpha_1}e_2^{\alpha_2} \cdots e_n^{\alpha_n}$ *as independent units of the algebra* (t having the same value as in Sec. 2). We denote these units by e_{mi}.

Now if we write w^m in terms of the units so defined, we get

$$w^m = \sum_{i=1}^{t} f_{mi}(x_1, x_2, \cdots x_n) e_{mi}, \qquad (m = 0, 1, \cdots).$$

It is easily shown that the coefficients of the units, f_{mi} are linearly inde-

pendent. They will also satisfy the differential equation $P \cdot U = 0$, since w^m satisfies the equation identically, by virtue of (1). Hence they may be taken as a fundamental set of solutions of degree m. Therefore

$$\sum_{m=0}^{\infty} \sum_{i=1}^{t} f_{mi} C_{mi},$$

where the C's are arbitrary constants, is a general solution of the differential equation, valid near the origin. But this is just the expansion of $(1 - w)^{-1}$ with the e's replaced by arbitrary constants. There follows the theorem:

If w is defined by conditions (a), (b), (c) and (d), any solution of the equation $P \cdot U = 0$ may be expressed as $(1 - w)^{-1}$ (or any other particular function of w whose expansion in powers of w contains all powers and with real coefficients) with the units replaced by suitable constants.

It is evident that the method can be extended at once to a system of equations in one dependent variable. Eq. 1 will be replaced by several equations, and the number of independent units of a particular degree in the generators will be correspondingly reduced. The method may also be extended to systems in more than one dependent variable.

As a simple example of the method in this section, we may consider the first order equation

$$\partial w/\partial x + 2\partial w/\partial y - \partial w/\partial z = 0.$$

The auxiliary equation is

$$e_1 + 2e_2 - e_3 = 0.$$

Taking e_1 and e_2 as independent units, e_3 is thus expressed in terms of them, and the hypervariable becomes

$$w = (x + z)e_1 + (y + 2z)e_2.$$

Following our general theorem we consider the series $\sum w^m$. Since every product of e_1 and e_2 is linearly independent of the others, and is thus a unit, when the units are all replaced by arbitrary constants there is obtained merely a general power series in the variables $x + z$ and $y + 2z$. Hence we obtain the usual general solution

$$F(x + z, y + 2z)$$

where F is an arbitrary analytic function. It is easily seen that for every first order equation the solution is expressible in a similar form.

As a second example, consider the equation

$$\partial^3 w/\partial x \partial y \partial z = 0.$$

The auxiliary equation is $e_1 e_2 e_3 = 0$. Hence all products of order 3 or more

including all three generators are zero. The algebra is the direct sum * of three subalgebras, whose units are the various products of e_1 and e_2, e_1 and e_3, and of e_2 and e_3 respectively, all these products being independent. Forming a power series in w, we see that no terms involving x, y, and z will appear, but every term involving any two of them will be present. We are thus led to the usual solution

$$F_1(x, y) + F_2(y, z) + F_3(x, z)$$

where F_1, F_2, F_3 are arbitrary analytic functions.

Whenever the differential equation is such that the auxiliary equation has two different real factors, the algebra will be the direct sum of two corresponding subalgebras, and the general solution will be the sum of two arbitrary functions of the separate hypervariables. If a factor is repeated there will be corresponding nilpotent elements in the algebra.

4. *Special forms of the algebra associated with a differential equation.* If we exclude differential equations of the form

$$(\partial^n/\partial x_1\, \partial x_2 \cdot \cdot \cdot \partial x_n) \cdot U = 0,$$

it will be possible to replace one of the generators by the unit 1 in the auxiliary equation without reducing its degree. Suppose this can be done with e_1. We impose the same conditions (a), (b), (c), and (d) on the algebra, there being in this case, of course, only $n-1$ generators $e_2, e_3, \cdot \cdot \cdot e_n$. Hence we have the same algebra as before with n replaced by $n-1$. But w^m will now depend not only on the units for which the sum of the exponents of the generators is m, but on all the units for which the sum is less than or equal to m, thus:

$$w^m = \sum_{j=0}^{m} \sum_{i=1}^{s} f_{mji}(x_1, x_2, \cdot \cdot \cdot x_n) e_{ji}, \qquad (m = 0, 1, 2, \cdot \cdot \cdot),$$

where $s = q_{n-1}(j) - q_{n-1}(j-r)$. Therefore for each m there will be t functions f_{mji}, all linearly independent. Hence the functions f_{mji} may be taken as a fundamental set of solutions of degree m of $P \cdot U = 0$.

Therefore

$$\sum_{m=0}^{\infty} \sum_{j=0}^{m} \sum_{i=1}^{s} f_{mji} C_{mji},$$

where the C's are arbitrary constants, is a general solution of the differential equation, valid near the origin.

For example, consider the equation

$$\partial^2 u/\partial x \partial y - \partial^2 u/\partial y^2 = 0.$$

* Dickson, *Algebras and Their Arithmetics*, p. 33.

The auxiliary equation is

$$e_1e_2 - e_2^2 = 0.$$

e_2 cannot be taken as the principal unit 1, because in that case the equation $e_1 - 1 = 0$ is of degree one while the auxiliary equation is of degree two. On the other hand if e_1 is replaced by 1, we get $e_2 - e_2^2 = 0$, as the equation to be satisfied by e_2. The hypervariable is

$$w = x + ye_2.$$

The algebra is finite, containing only the two units 1 and e_2. On forming w^m,

$$w^m = x^m + [(x+y)^m - x^m]e_2.$$

From the above theorem the general solution is thus obtained in the form

$$u = f(x) + g(x+y) - g(x)$$

which can, of course, be condensed into the form

$$u = F(x) + g(x+y)$$

where f, g and F are arbitrary analytic functions.

For the equations which will be treated in the succeeding sections it is possible to transform the algebra so that the square of each unit has a component in the direction of the principal unit $e_1 = 1$, but no product of two different units involves e_1. In this case, for any power series in w, if

$$\alpha_m = \sum_{j=0}^{m} \sum_{i=1}^{s} a_{mji}e_{ji} \quad \text{and if} \quad e_{ji}^2 = \gamma_{ji}e_1 + \cdots,$$

then

$$\sum_{m=0}^{\infty} \alpha_m w^m = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \sum_{i=1}^{s} a_{mji}f_{mji}\gamma_{ji}e_1 + \cdots.$$

Hence *the real part* (the coefficient of the principal unit e_1), *of a power series in w is a general solution of the differential equation,* valid near the origin.

Thus far we have considered only the formal connection between the general solution of a differential equation and its algebra. The theoretical value of the ordinary complex variable z in the case of Laplace's equation in two variables lies in the theorem that any solution of the equation is the real part of some *analytic* function of z.

The proof consists in showing that to every function u which satisfies Laplace's equation there corresponds a function v such that u and v satisfy the Cauchy-Riemann equations. The equations which correspond to the Cauchy-Riemann equations for a hypervariable (that is, which express the condition for a unique differential coefficient) were first obtained by Scheffers.*

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* *Leipziger Berichte*, Vol. 45 (1893), p. 828.

For an algebra with an infinity of units the Scheffers equations are also infinite in number, and it is difficult to treat such general cases. But in special cases, in particular those cases considered in the succeeding sections, it is possible to prove as in complex variables, that given any solution of a differential equation, other solutions can be determined which satisfy the Scheffers equations of the corresponding hypervariable, and thus form an analytic function of the hypervariable.

5. *Applications to particular equations.* From the original definition of t we see that it is independent of m if and only if there are two independent variables. When t is a function of m the number of coefficients f_{mji} for w^m increases with m, and the algebra will contain an infinity of units. Thus, for two independent variables there is a finite algebra, but for more than two independent variables there is an infinity of units.

Laplace's equation in two variables is a very special equation because t is 2 and the number of units equals the number of variables. The number of units can equal the number of variables only in the case of a second order equation in two variables.

For Laplace's equation in three variables, $t = 2m + 1$. The corresponding hypervariable has already been defined above.

For Airy's fundamental equation of elasticity, namely

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2)^2 \cdot U = 0,$$

t is 4. The variable

$$w = x + ye_3 \quad \text{where} \quad e_3{}^2 = 1 + 2ie_3, \quad [i = (-1)^{1/2}],$$

has units satisfying the conditions of the last section. This algebra containing the four units $e_1 = 1$, $e_2 = i$, e_3, $e_4 = ie_3$, is the simplest complex nilpotent algebra, since the square of $e_3 - i$ vanishes. Calling this nilpotent number E, the hypervariable may be written

$$w = x + iy + yE = z + yE$$

where z is an ordinary complex variable. Then

$$w^m = z^m + myz^{m-1}E,$$

and

$$\begin{aligned} f(w) &= f(z) + yf'(z)E \\ &= f(z) - yf'(z)i + yf'(z)e_3 \end{aligned}$$

and the real part of $f(w)$, that is, the real part of $f(z)$ plus the imaginary part of $yf'(z)$, is the general solution of the differential equation. The Scheffers equations are obtained by writing

$$f(w) = f_1 + f_2 i + f_3 e_3 + f_4 i e_3,$$

the f's being functions of x and y, and finding the conditions that there exist a function $f'(w)$ such that

$$df = f'(w)\,dw$$

for all values of dw. The equations are

$$\partial f_1/\partial y = \partial f_3/\partial x, \qquad \partial f_3/\partial y = \partial f_1/\partial x - 2\partial f_4/\partial x,$$
$$\partial f_2/\partial y = \partial f_4/\partial x, \qquad \partial f_4/\partial y = \partial f_2/\partial x + 2\partial f_3/\partial x.$$

On elimination between these equations, it is seen that the functions f_1, f_2, f_3, f_4 satisfy Airy's equation, and conversely it is easily shown that given any solution f_1 of Airy's equation, functions f_2, f_3, and f_4 can be determined so that the Scheffers equations are satisfied. Hence *the real part, f_1, of an arbitrary analytic function of w forms a general solution of Airy's equation.*

6. *Laplace's equation in n variables.* Let

$$\Delta_n = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2},$$

where $x_1, x_2, \cdots x_n$ are ordinary complex variables. The hypervariable connected with the equation $\Delta_2 \cdot U = 0$ is $x_1 + ix_2$, and for $\Delta_3 \cdot U = 0$ is

$$x_1 + ix_2 \cos\alpha + ix_3 \sin\alpha,$$

where $\cos\alpha$ and $\sin\alpha$ are units. Thus to pass from the case $n = 2$ to $n = 3$ we have substituted the hypervariable $x_2 \cos\alpha + x_3 \sin\alpha$ for x_2. We shall show that this is true in general, so that the hypervariable for $\Delta_4 \cdot U = 0$ (the wave equation) is

$$w = x_1 + ix_2 \cos\alpha + ix_3 \sin\alpha \cos\beta + ix_4 \sin\alpha \sin\beta, \text{ etc.}$$

It will be shown that *any solution of $\Delta_n \cdot U = 0$ is the scalar part* (ordinary complex coefficient of the principal unit 1) *of some analytic function of the corresponding hypervariable.*

For values of n larger than 3 the Scheffers equations become too complicated to integrate directly, so the proof will be effected by a process of induction. For simplicity of notation the induction will only be carried from $n = 3$ to $n = 4$, but it will be seen that the method is general.

Let $$\omega = x_3 \cos\beta + x_4 \sin\beta.$$

We first transform ω by writing $\cos\beta$ and $\sin\beta$ as exponentials. This means only a linear transformation of the units of the algebra:

$$\omega = ue^{i\beta} + ve^{-i\beta} \equiv \omega',$$

where

$$2u = x_3 - ix_4, \qquad 2v = x_3 + ix_4.$$

Transforming the differential equation gives

$$\Delta_4 \cdot U = (\Delta_2 + \partial^2/\partial u\, \partial v) \cdot U \equiv \Delta'_4 \cdot U.$$

Let

$$F(\omega') = \sum_{-\infty}^{\infty} F_k(u, v) e^{ki\beta}.$$

The necessary and sufficient condition that $F(\omega')$ be analytic, is that the Scheffers equations for ω':

$$\partial F_{k+1}/\partial u = \partial F_{k-1}/\partial v = F'_k, \tag{2}$$

where

$$F'(\omega') = \sum F'_k(u, v) e^{ki\beta},$$

should be satisfied.

Now consider any two solutions $f_{00}(x_1, x_2, u, v)$ and $f_{01}(x_1, x_2, u, v)$ of the equation $\Delta'_4 \cdot U = 0$. Calling $F_k = f_{0k}$, a set of functions f_{0k} can be determined in terms of f_{00} and f_{01} which will at the same time be solutions of $\Delta'_4 \cdot U = 0$ and also satisfy equations (2).

To prove this, consider the equations for $k = 2$:

$$\partial f_{02}/\partial u = \partial f_{00}/\partial v, \tag{3}$$

$$\partial^2 f_{02}/\partial x_1^2 + \partial^2 f_{02}/\partial x_2^2 + \partial^2 f_{02}/\partial u \partial v = 0. \tag{4}$$

Differentiating (3) and substituting in (4) gives

$$\partial^2 f_{02}/\partial x_1^2 + \partial^2 f_{02}/\partial x_2^2 + \partial^2 f_{00}/\partial v^2 = 0. \tag{5}$$

(3) and (5) are equivalent to (3) and (4). Now f_{02} determined by (3) leaves $f_{02}(x_1, x_2, u_0, v)$ arbitrary, u_0 being a particular value of u. But since (5) contains no u derivatives, if it is satisfied for a particular value of u, say u_0, then it will be satisfied for all values of u. Hence if $f_{02}(x_1, x_2, u_0, v)$ is determined by means of (5), f_{02} and its x_1 derivative still being arbitrary for $x_1 = x_{10}$ and $u = u_0$, f_{02} thus found will satisfy (3) and (5) identically.

We may thus determine successively the functions f_{0k} in terms of f_{00} and f_{01}. In each case f_{0k} and its x_1 derivative are arbitrary at $x_1 = x_{10}$ and $u = u_0$ for $k > 0$, and at $x_1 = x_{10}$ and $v = v_0$ for $k < 0$.

Hence

$$f_0(x_1, x_2, \omega') \equiv \sum_{-\infty}^{\infty} f_{0k}(x_1, x_2, u, v) e^{ki\beta}$$

will be an analytic function of x_1, x_2, and ω'. Now we have

$$\partial^2 f_0/\partial x_1^2 = \sum (\partial^2 f_{0k}/\partial x_1^2) e^{ki\beta}, \tag{6}$$

and a similar relation (7) with respect to x_2. Since the derivative of an analytic function of ω' is itself analytic, from (2)

$$\partial^2 f_0/\partial \omega'^2 = \sum (\partial^2 f_{0k}/\partial u \partial v) e^{ki\beta}. \tag{8}$$

Adding (6), (7) and (8), since f_{0k} are solutions of $\Delta'_4 \cdot U = 0$,

$$\partial^2 f_0/\partial x_1^2 + \partial^2 f_0/\partial x_2^2 + \partial^2 f_0/\partial \omega'^2 = 0.$$

The problem has thus been reduced one stage. We already know that given an analytic function $f_0(x_1, x_2, \omega')$ satisfying this equation there can be found functions f_m such that $\sum f_m e^{mia}$ is an analytic function of

$$x_1 + ix_2 \cos\alpha + i\omega' \sin\alpha.*$$

It has thus been proved that any function $f_{00}(x_1, x_2, u, v)$ which satisfies $\Delta'_4 \cdot U = 0$ is the scalar part of some analytic function $f(w')$ of

$$w' = x_1 + ix_2 \cos\alpha + iu \sin\alpha e^{i\beta} + iv \sin\alpha e^{-i\beta}.$$

Now suppose $F_{00}(x_1, x_2, x_3, x_4)$ is any solution of $\Delta_4 \cdot U = 0$. Expressing x_3 and x_4 in terms of u and v we get a function $f_{00}(x_1, x_2, u, v)$ which satisfies $\Delta_4' \cdot U = 0$. By the above theorem this function will be the scalar part of a function $f(w')$. If $f(w')$ and w' are transformed by writing u and v in terms of x_3 and x_4, and changing the exponentials back to sines and cosines, we obtain w and $F(w)$. The scalar part remains unchanged under this transformation, so that F_{00} will be the scalar part of $F(w)$. Hence we obtain the final theorem that any solution of $\Delta_4 \cdot U = 0$ is the scalar part of some analytic function of w. This is, of course, the wave equation with $x_1 = it$. *For the ordinary form of the wave equation:*

$$\Delta_3 U - \partial^2 U/\partial t^2 = 0,$$

the hypervariable is

$$w = t + x_1 e_{10} + x_2 e_{-11} + x_3 e_{-1-1}$$

where the double infinity of units e_{ij} has a multiplication table determined by the relations

$$e_{\pm i \pm j} = \begin{matrix}\cos \\ \sin\end{matrix} i\alpha \begin{matrix}\cos \\ \sin\end{matrix} j\beta.$$

For example,

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* This was of course proved only for the case where ω' was real or complex. But the proof depends only on the fact that the functions are analytic, so the results may be generalized at once to any hypervariable ω'.

$$e_{-1-4}e_{-21} = \sin\alpha \sin 4\beta \sin 2\alpha \cos\beta$$
$$= \tfrac{1}{4}(\cos\alpha \sin 5\beta + \cos\alpha \sin 3\beta - \cos 3\alpha \sin 5\beta - \cos 3\alpha \sin 3\beta)$$
$$= \tfrac{1}{4}(e_{1-5} + e_{1-3} - e_{3-5} - e_{3-3}).$$

Also, $w^2 = (t + \frac{1}{2}x_1^2 + \frac{1}{4}x_2^2 + \frac{1}{4}x_3^2) + 2x_1te_{10}$
$+ (\frac{1}{2}x_1^2 - \frac{1}{4}x_2^2 - \frac{1}{4}x_3^2)e_{20} + \frac{1}{4}(x_2^2 - x_3^2)e_{02}$
$+ \frac{1}{2}x_2x_3e_{0-2} + 2x_2te_{-11}' + 2x_3te_{-1-1} - \frac{1}{2}x_2x_3e_{2-2}$
$+ x_1x_2e_{-21} + x_1x_3e_{-2-1} + \frac{1}{4}(-x_2^2 + x_3^2)e_{22}.$

Each of the coefficients of the units satisfies the wave equation. These coefficients and the coefficients for the higher powers are *natural generalizations of spherical harmonics to the four dimensional space* of special relativity in which distance is given by the formula

$$s^2 = t^2 - x_1^2 - x_2^2 - x_3^2.$$

In a similar way we can generalize spherical harmonics to n dimensions.

7. *Simultaneous partial differential equations.* The two conjugate functions of the analytic function of the ordinary complex variable z constitute a general solution of the Cauchy-Riemann equations. If the real part of $f(z)$ is taken as the x component and the imaginary part as the negative of the y component of a vector field, the Cauchy-Riemann equations express the fact that the curl and divergence of the field vanishes.

In three dimensions the condition that the curl and divergence of a vector field vanish gives the following linear equations on the three components α, β and γ.*

$$\partial\gamma/\partial y = \partial\beta/\partial z, \qquad \partial\beta/\partial x = \partial\alpha/\partial y,$$
$$\partial\alpha/\partial z = \partial\gamma/\partial x, \qquad \partial\alpha/\partial x + \partial\beta/\partial y + \partial\gamma/\partial z = 0.$$

In physics these equations play a rôle intermediate between the Cauchy-Riemann and Maxwell's equations.

If w is the previously defined hypervariable belonging to Laplace's equation in three variables, and if $f(w) = \Sigma f_k e_k$ is any analytic function of w, then *a general solution of the four equations on α, β and γ is obtained by putting*

$$(9) \qquad \alpha = f_0, \qquad \beta = -f_1/2, \qquad \gamma = -f_{-1}/2.$$

That is, for any three functions α, β and γ satisfying these four equations, functions f_k can be found satisfying (9) and such that $\Sigma f_k e_k$ is an analytic function of w.

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* See Hanni, *Tôhoku Mathematical Journal*, Vol. 5 (1914), p. 145.

A REMAINDER FOR THE EULER-MacLAURIN SUMMATION FORMULA IN TWO INDEPENDENT VARIABLES.*

By WILLIAM DOWELL BATEN.

The object of this article is to develop a remainder for the Euler-MacLaurin summation formula in two independent variables. The first part defines and develops Bernoulli product polynomials for two independent variables. The second part gives a derivation of Euler-MacLaurin summation formula for two variables by employing certain properties of the polynomials obtained in the first part. A remainder is developed in terms of definite integrals and in terms of derivatives.

Conditions for convergence are given when the summation extends to infinity for both variables.

Krause studied this problem and developed a remainder for this summation formula in two variables for functions which can be expressed in a Taylor series. It appears that he was not satisfied with the applicability of his remainder term. The remainder given in this article is much less complicated and more practical than that obtained by Krause.† The remainder for the summation formula for two variables is not as simple as that given by Steffensen for one variable.‡

1. *Bernoulli product polynomials.* Let $B_{m,n}(x, y)$ be a polynomial in two independent variables x and y such that

$$\Delta_x \Delta_y B_{m,n}(x, y) = mnx^{m-1}y^{n-1}, \quad \text{and} \tag{1}$$

$$D_x{}^i D_y{}^j B_{m,n}(x, y) = m(i)n(j)B_{m-i,n-j}(x, y),\S \tag{2}$$

where Δ_x and Δ_y represent finite differences, $D_z{}^s$ represents the s-th derivative with respect to z, and the first subscript of B denotes the degree of the polynomial in x while the second denotes the degree of the polynomial in y.

By Taylor's theorem

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* Presented to the Society December 31, 1930, at Cleveland.

† Krause, "Über Bernoullische Zahlen und Funktionen im Gebiete der Funktionen zweier veränderlichen Grossen," *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig*, Vol. 55-56 (1903), pp. 39-62.

‡ Steffensen, *Interpolation*, pp. 119-138.

§ $m(r) = m(m-1)(m-2)\cdots(m-r+1) = (m!)/(m-r)!$.

$$\begin{aligned} B_{m,n}(x+h, y+k) = {} & B_{m,n}(x,y) + hD_xB_{m,n}(x,y) + kD_yB_{m,n}(x,y) \\ & + (1/2)\,(h^2D_x^2B_{m,n}(x,y) + 2hkD_xD_yB_{m,n}(x,y) \\ & + k^2D_y^2B_{m,n}(x,y)) \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ = {} & \sum_{j=0}^{n} \sum_{i=0}^{m} (1/j!)\,(1/i!)\,h^ik^jD_x^iD_y^jB_{m,n}(x,y). \end{aligned}$$

By property (2)

$$B_{m,n}(x+h, y+k) = \sum_{j=0}^{n} \sum_{i=0}^{m} \binom{m}{i} \binom{n}{j} h^ik^jB_{m-i,n-j}(x,y).$$

By writing $m - i$ for i and $n - j$ for j the above becomes

$$(3) \qquad B_{m,n}(x+h, y+k) = \sum_{j=0}^{n} \sum_{i=0}^{m} \binom{m}{i} \binom{n}{j} h^{m-i}k^{n-j}B_{i,j}(x,y).$$

By use of (3) for h and k equal to 1 and by (1)

$$B_{m,n}(x+1, y+1) - B_{m,n}(x+1, y) - B_{m,n}(x, y+1) + B_{m,n}(x,y)$$

$$= \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} \binom{m}{i} \binom{n}{j} B_{i,j}(x,y) = \Delta_x\Delta_yB_{m,n}(x,y) = mnx^{m-1}y^{n-1}.$$

Or

$$(4) \qquad \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} \binom{m}{i} \binom{n}{j} B_{i,j}(x,y) = mnx^{m-1}y^{n-1}.$$

For various values of m and n these polynomials become

$$(5) \quad \begin{cases} B_{0,0}(x,y) = 1, \\ B_{1,0}(x,y) = x - 1/2, \qquad B_{0,1}(x,y) = y - 1/2, \\ B_{1,1}(x,y) = xy - x/2 - y/2 + 1/4, \\ B_{2,1}(x,y) = x^2y - xy - x^2/2 + x/2 + y/6 - 1/12, \\ B_{1,2}(x,y) = xy^2 - xy - y^2/2 + y/2 + x/6 - 1/12, \\ B_{2,0}(x,y) = x^2 - x + 1/6, \qquad B_{0,2}(x,y) = y^2 - y + 1/6, \\ B_{2,2}(x,y) = x^2y^2 - xy^2 - x^2y + xy + x^2/6 + y^2/6 - x/6 - y/6 + 1/36, \\ \text{Etc.} \end{cases}$$

From the above polynomials it is seen that

$$B_{i,j}(x,y) = B_{i,0}(x,y) \cdot B_{0,j}(x,y),$$

where i and j run from 0 to 2. This is evident for any i and j from (1) and (2) since x and y are independent. This shows that these polynomials for two variables x and y are the products of Bernoulli polynomials for the single variables. $B_{i,0}(x,y)$ is a Bernoulli polynomial in the single variable x while $B_{0,j}(x,y)$ is a Bernoulli polynomial in the single variable y. The polynomials in the two variables will be called Bernoulli product polynomials.

The values of the product polynomials for $x = y = 0$ will be called Bernoulli product numbers, and will be represented by the symbol

$$B_{m,n}$$

where m and n take various values. These Bernoulli product numbers may be obtained from equations 5 by substitution of zero for x and y. But they may be treated independently by use of (4) for $x = y = 0$ as

$$B_{0,0} = 1; \quad \sum_{j=0}^{n} \sum_{i=0}^{m} \binom{m}{i} \binom{n}{j} B_{i,j} = B_{m,n}, \qquad (m, n > 1).$$

This may be written in a symbolical form as

$$\{(B_{0,0} + 1)^m - B_{m,0}\}\{(B_{0,0} + 1)^n - B_{0,n}\},$$

after the expansion the exponents of the first brackets are changed to the first subscript of the B's, while the exponents of the second brackets are changed to the second subscript of the B's.

Bernoulli product polynomials may be expressed explicitly by Bernoulli product numbers by substituting in (3), $x = y = 0$, and then setting $h = x$ and $k = y$. This becomes

$$B_{m,n}(x, y) = \sum_{j=0}^{n} \sum_{i=0}^{m} \binom{m}{i} \binom{n}{j} B_{i,j} x^{m-i} y^{n-j}.$$

This may be written symbolically as

$$B_{m,n}(x, y) = (x + B)^m (y + B)^n.$$

These Bernoulli product polynomials are not the same as the Bernoulli polynomials in two variables defined by Appell,* or the Bernoulli functions in the region of two independent variables developed by Krause,† or the double Bernoulli polynomials treated by Barnes.‡ The product numbers are not the same as the Bernoulli numbers in the region of two independent variables treated by Krause or the Bernoulli double numbers defined by Barnes. These differ from the Bernoulli polynomials in two variables mentioned by Nörlund.§

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* P. Appell, "Sur les functions de Bernoulli à deux variables," *Archiv der Mathematik und Physick* (3), Vol. 4 (1902), pp. 292-3.

† Krause, "Über Bernoullische Zahlen und Funktionen im Gebiete der Funktionen zweier veränderlichen Grossen," *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig*, Vol. 55-56 (1903), pp. 39-62.

‡ Barnes, "The Theory of the Double Gamma Functions," *Philosophical Transactions*, London, 196 A (1901), pp. 271-278.

§ Nörlund, "Mémoire sur les polynomials de Bernoulli," *Acta Mathematica*, Vol. 43, p. 121.

2. *Euler-MacLaurin Summation Formula in two Variables.* Let $\bar{B}_{m,n}(x,y)$ be a periodic function of period 1 with respect to x and y such that $\bar{B}_{m,n}(x,y)$ is identical with $B_{m,n}(x,y)/m!\,n!$ for $0 \leqq x < 1$ and $0 \leqq y < 1$. This function is determined for all values of x and y, by having to satisfy the conditions

$$\bar{B}_{m,n}(x,y) = B_{m,n}(x,y)/m!\,n! \quad \text{for} \quad 0 \leqq x < 1 \quad \text{and} \quad 0 \leqq y < 1,$$
$$\bar{B}_{m,n}(x+1, y+1) = \bar{B}_{m,n}(x,y) \quad \text{for all } x \text{ and all } y.$$

If m or n is not equal to 1, $\bar{B}_{m,n}(x,y)$ is continuous for all x and for all y. But when m or n is equal to 1 this periodic function becomes discontinuous at the points (r,s), where r and s are integers. $\bar{B}_{m,n}(x,y)$ is continuous in x if m is not equal to 1 and continuous in y if n is not equal to 1. Hence

$$D_x^i D_y^j \bar{B}_{m,n}(x,y) = \bar{B}_{m-i,n-j}(x,y), \quad \text{for} \quad m-i,\ n-j > 0.$$

Hence $\bar{B}_{m,n}(x,y)$ possesses continuous differential coefficients of x and of y of all orders up to $m-2$ for x and up to $n-2$ for y, while the differential coefficients of order $m-1$ for x and $n-1$ for y are discontinuous at the points (r,s), where r and s are integers, and also for the points where either r or s is an integer. Yet $\bar{B}_{m,1}(x,y)$ has continuous derivatives of all orders up to $m-2$ in x, while $\bar{B}_{1,n}(x,y)$ has continuous derivatives of all orders up to $n-2$ in y.

Consider the double integral

$$R_{m,n} = \int_0^1 \int_0^1 \bar{B}_{m,n}(\Theta - t, \Psi - k) f_{m,n}(x+t, y+k)\,dt\,dk,$$

where the m in the subscript of the function f represents the m-th derivative with respect to t and the n in the subscript of f means the n-th derivative of f with respect to k. f is a continuous function in x and y and possesses continuous derivatives of all orders in x and y, which are integrable.

Integrate by parts with respect to t and then with respect to k

$$R_{m,n} = \bar{B}_{m,n}(\Theta, \Psi)\Delta_x\Delta_y f_{m-1,n-1}(x,y) + P_{m,n-1} + R_{m-1,n},$$

where

$$P_{m,n-1} = \int_0^1 \bar{B}_{m,n-1}(\Theta, \Psi - k)\Delta_x f_{m-1,n-1}(x, y+k)\,dk.$$

By repeating this for $R_{m-1,n}$, $R_{m-2,n}$, etc.,

$$R_{m,n} = \sum_{v=2}^{m} \bar{B}_{v,n}(\Theta, \Psi)\Delta_x\Delta_y f_{v-1,n-1}(x,y) + \sum_{v=2}^{m} P_{v,n-1} + R_{1,n}.$$

$$P_{m,n-1} = \bar{B}_{m,n-1}(\Theta, \Psi)\Delta_x\Delta_y f_{m-1,n-2}(x,y) + P_{m,n-2}.$$

By continuing this process for the P's

$$P_{m,n-1} = \sum_{s=2}^{n-1} \bar{B}_{m,s}(\Theta, \Psi)\Delta_x\Delta_y f_{m,s-1}(x, y) + P_{m,1}.$$

$$P_{m,1} = \int_0^1 \bar{B}_{m,1}(\Theta, \Psi - k)\Delta_x f_{m-1,1}(x, y + k)\,dk.$$

Now it is assumed that

$$0 \leqq \Theta \leqq 1, \qquad 0 \leqq \Psi \leqq 1.$$

From the nature of $\bar{B}_{m,n}$ and $B_{m,n}$, $\bar{B}_{m,n} = \bar{B}_{m,0} \cdot \bar{B}_{0,n}$. As $B_{0,1}(\Theta, \Psi - k)$ is discontinuous for $k = \Psi$, $P_{m,1}$ can be written in the form

$$P_{m,1} = (\bar{B}_{m,0})\{\int_0^\Psi \bar{B}_{0,1}(\Theta, \Psi - k)\Delta_x f_{m-1,1}(x, y + k)\,dk + \int_\Psi^1 \bar{B}_{0,1}(\Theta, \Psi - k + 1)\Delta_x f_{m-1,1}(x, y + k)\,dk\},$$

whence, as $B_{0,1}(x, y) = y - 1/2$ for $0 \leqq y < 1$,

$$\begin{aligned} P_{m,1} &= (\bar{B}_{m,1})\{-\int_0^1 k\Delta_x f_{m-1,1}(x, y + k)\,dk \\ &\quad + (\Psi - 1/2)\int_0^1 \Delta_x f_{m-1,1}(x, y + k)\,dk + \int_\Psi^1 \Delta_x f_{m-1,1}(x, y + k)\,dk\} \\ &= (\bar{B}_{m,1})\{-\Delta_x f_{m-1,0}(x, y + \Psi) + \bar{B}_{0,1}(\Theta, \Psi)\Delta_x\Delta_y f_{m-1,0}(x, y) \\ &\quad + \int_y^{y+1} \Delta_x f_{m-1,0}(x, k)\,dk\}. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{v=2}^{m} P_{v,n-1} &= \sum_{v=2}^{m}\sum_{s=1}^{n-1} \bar{B}_{v,s}(\Theta, \Psi)\Delta_x\Delta_y f_{v-1,s-1}(x, y) \\ &\quad - \sum_{m}^{v=2} \bar{B}_{v,0}(\Theta, \Psi)\Delta_x f_{v-1,0}(x, y + k) + \sum_{v=2}^{m} \bar{B}_{v,0}\int_y^{y+1} \Delta_x f_{m-1,0}(x, k)\,dk. \end{aligned}$$

$$\begin{aligned} R_{1,n} &= \int_0^1\int_0^1 \bar{B}_{1,n}(\Theta - t, \Psi - k) f_{1,n}(x + t, y + k)\,dt\,dk \\ &= \int_0^1 \bar{B}_{1,n}(\Theta - t, \Psi)\Delta_y f_{1,n-1}(x + t, y)\,dt + R_{1,n-1} \\ &= \bar{B}_{0,n}(\Theta, \Psi)\int_0^1 \bar{B}_{1,0}(\Theta - t, \Psi)\Delta_y f_{1,n-1}(x + t, y)\,dt + R_{1,n-1} \\ &= (\bar{B}_{0,n}(\Theta, \Psi))\{-\Delta_y f_{0,n-1}(x + \Theta, y) + \bar{B}_{1,0}(\Theta, \Psi)\Delta_x\Delta_y f_{0,n-1}(x, y) \\ &\quad + \int_x^{x+1} \Delta_y f_{0,n-1}(t, y)\,dt\} + R_{1,n-1} \end{aligned}$$

$$= -\bar{B}_{0,n}(\Theta,\Psi)\Delta_y f_{0,n-1}(x+\Theta,y)+\bar{B}_{1,n}(\Theta,\Psi)\Delta_x\Delta_y f_{0,n-1}(x,y)$$
$$+\bar{B}_{0,n}(\Theta,\Psi)\int_x^{x+1}\Delta_y f_{0,n-1}(t,y)\,dt+R_{1,n-1}.$$

Therefore

$$R_{1,n} = -\sum_{s=1}^{n}\bar{B}_{0,s}(\Theta,\Psi)\Delta_y f_{0,s-1}(x+\Theta,y)+\sum_{s=1}^{n}\bar{B}_{1,s}(\Theta,\Psi)\Delta_x\Delta_y f_{0,s-1}(x,y)$$
$$+\sum_{s=1}^{n}\bar{B}_{0,s}(\Theta,\Psi)\int_x^{x+1}\Delta_y f_{0,s-1}(t,y)\,dt+f(x+\Theta,y+\Psi)$$
$$-\bar{B}_{1,0}(\Theta,\Psi)\Delta_x f_{0,0}(x,y+\Psi)-\int_x^{x+1}f(t,y+\Psi)\,dt-\int_y^{y+1}f(x+\Theta,k)\,dk$$
$$-\bar{B}_{1,0}\int_y^{y+1}\Delta_x f(x,k)\,dk+\int_x^{x+1}\int_y^{y+1}f(t,k)\,dt\,dk,$$

after $R_{1,1}$ is treated similarly to $P_{m,1}$. Therefore

$$R_{m,n} = \sum_{v=1}^{m}\sum_{s=1}^{n}\bar{B}_{v,s}(\Theta,\Psi)\Delta_x\Delta_y f_{v-1,s-1}(x,y)-\sum_{v=1}^{m}\bar{B}_{v,0}(\Theta,\Psi)\Delta_x f_{v-1,0}(x,y+\Psi)$$
$$-\sum_{s=1}^{n}\bar{B}_{0,s}(\Theta,\Psi)\Delta_y f_{0,s-1}(x+\Theta,y)+\sum_{v=1}^{m}\bar{B}_{v,0}(\Theta,\Psi)\int_y^{y+1}\Delta_x f_{v-1,0}(x,y)\,dy$$
$$+\sum_{s=1}^{n}\bar{B}_{0,s}(\Theta,\Psi)\int_x^{x+1}\Delta_y f_{0,s-1}(t,y)\,dt-\int_x^{x+1}f(t,y+\Psi)\,dt$$
$$-\int_y^{y+1}f(x+\Theta,k)\,dk+\int_x^{x+1}\int_y^{y+1}f(t,k)\,dt\,dk+f(x+\Theta,y+\Psi).$$

Find the sum from $x=0$ to $w-1$ and then from $y=0$ to $L-1$. This becomes after transposing $f(x+\Theta,y+\Psi)$ and $R_{m,n}$

$$(6)\quad \sum_{y=0}^{L-1}\sum_{x=0}^{w-1}f(x+\Theta,y+\Psi) = -\sum_{v=1}^{m}\sum_{s=1}^{n}\bar{B}_{v,s}(\Theta,\Psi)f_{v-1,s-1}(x,y)\Big]_0^L\Big]_0^w$$
$$-\sum_{s=1}^{n}\bar{B}_{0,s}(\Theta,\Psi)\int_0^w f_{0,s-1}(t,y)\,dt\Big]_0^L$$
$$-\sum_{v=1}^{m}\bar{B}_{v,0}(\Theta,\Psi)\int_0^L f_{v-1,0}(x,y)\,dy\Big]_0^w$$
$$+\sum_{v=1}^{m}\sum_{y=0}^{L-1}\bar{B}_{v,0}(\Theta,\Psi)f_{v-1,0}(x,y+k)\Big]_0^w$$
$$+\sum_{x=0}^{w-1}\sum_{s=1}^{n}\bar{B}_{0,s}(\Theta,\Psi)f_{0,s-1}(x+\Theta,y)\Big]_0^L$$
$$+\sum_{y=0}^{L-1}\int_0^w f(t,y+\Psi)\,dt+\sum_{x=0}^{w-1}\int_0^L f(x+\Theta,k)\,dk$$
$$-\int_0^w\int_0^L f(t,k)\,dk\,dt-\sum_{y=0}^{L-1}\sum_{x=0}^{w-1}R_{m,n}.$$

Apply the Euler-MacLaurin summation formula for a single variable to the summations

$$\sum_{y=0}^{L-1} f_{v-1,0}(x, y+\Psi)\Big]_0^w, \qquad \sum_{x=0}^{w-1} f_{0,s-1}(x+\Theta, y)\Big]_0^L,$$

$$\sum_{y=0}^{L-1} \int_0^w f(t, y+\Psi)\,dt, \qquad \sum_{x=0}^{w-1} \int_0^L f(\Theta+x, k)\,dk;$$

these summations become

$$\sum_{y=0}^{L-1} \sum_{v=1}^{m} \bar{B}_{v,0}(\Theta,\Psi) f_{v-1,0}(x, y+\Psi)\Big]_0^w = \sum_{v=1}^{m} \bar{B}_{v,0}(\Theta,\Psi) \int_0^L f_{v-1,0}(x,k)\,dk\Big]_0^w$$
$$+ \sum_{v=1}^{m} \sum_{s=1}^{n} \bar{B}_{v,s}(\Theta,\Psi) f_{v-1,s-1}(x,y)\Big]_0^w \Big]_0^L$$
$$- \sum_{v=1}^{m} \int_0^L \bar{B}_{v,n}(\Theta,\Psi-k) f_{v-1,n}(x,k)\,dk\Big]_0^w,$$

$$\sum_{x=0}^{w-1} \sum_{s=1}^{n} \bar{B}_{0,s}(\Theta,\Psi) f_{0,s-1}(x+\Theta, y)\Big]_0^L = \sum_{s=1}^{n} \bar{B}_{0,s}(\Theta,\Psi) \int_0^w f_{0,s-1}(t,y)\,dt\Big]_0^L$$
$$+ \sum_{v=1}^{m} \sum_{s=1}^{n} \bar{B}_{v,s}(\Theta,\Psi) f_{v-1,s-1}(x,y)\Big]_0^L \Big]_0^w$$
$$- \sum_{s=1}^{n} \int_0^w \bar{B}_{m,s}(\Theta-t,\Psi) f_{m,s-1}(t,y)\,dt\Big]_0^L,$$

$$\sum_{y=0}^{L-1} \int_0^w f(t, y+k)\,dt = \int_0^L \int_0^w f(t,k)\,dt\,dk + \sum_{s=1}^{n} \bar{B}_{0,s}(\Theta,\Psi) \int_0^w f_{0,s-1}(t,y)\,dt\Big]_0^L$$
$$- \int_0^L \bar{B}_{0,n}(\Theta,\Psi-k) \int_0^w f_{0,n}(t,k)\,dt\,dk,$$

$$\sum_{x=0}^{w-1} \int_0^L f(x+\Theta, k)\,dk = \int_0^w \int_0^L f(t,k)\,dk\,dt + \sum_{v=1}^{m} \bar{B}_{v,0}(\Theta,\Psi) \int_0^L f_{v-1,0}(x,k)\,dk\Big]_0^w$$
$$- \int_0^w \bar{B}_{m,0}(\Theta-t,\Psi) \int_0^L f_{m,0}(t,k)\,dk\,dt.$$

Substituting these single summations in the double summation in (6) for $\sum_{y=0}^{L-1} \sum_{x=0}^{w-1} f(x+\Theta, y+\Psi)$, it becomes

$$\sum_{y=0}^{L-1} \sum_{x=0}^{w-1} f(x+\Theta, y+\Psi) = \int_0^L \int_0^w f(t,k)\,dt\,dk$$
$$+ \sum_{v=1}^{m} \sum_{s=1}^{n} \bar{B}_{v,s}(\Theta,\Psi) f_{v-1,s-1}(x,y)\Big]_0^L \Big]_0^w$$
$$+ \sum_{v=1}^{m} \bar{B}_{v,0}(\Theta,\Psi) \int_0^L f_{v-1,0}(x,k)\,dk\Big]_0^w$$

$$+\sum_{s=1}^{n}\bar{B}_{0,s}(\Theta,\Psi)\int_0^{w}f_{0,s-1}(t,y)\,dt\Big]_0^{L}$$

$$-\sum_{v=1}^{m}\int_0^{L}\bar{B}_{v,n}(\Theta,\Psi-k)f_{v-1,n}(x,k)\,dk\Big]_0^{w}$$

$$-\sum_{s=1}^{n}\int_0^{w}\bar{B}_{m,s}(\Theta-t,\Psi)f_{m,s-1}(t,y)\,dt\Big]_0^{L}$$

$$-\int_0^{L}\bar{B}_{0,n}(\Theta,\Psi-k)\int_0^{w}f_{0,n}(t,k)\,dt\,dk$$

$$-\int_0^{w}\bar{B}_{m,0}(\Theta-t,\Psi)\int_0^{L}f_{m,0}(t,k)\,dk\,dt\;-\sum_{y=0}^{L-1}\sum_{x=0}^{w-1}R_{m,n}.$$

$$\sum_{y=0}^{L-1}\sum_{x=0}^{w-1}R_{m,n}=\sum_{y=0}^{L-1}\sum_{x=0}^{w-1}\int_0^{1}\int_0^{1}\bar{B}_{m,n}(\Theta-t,\Psi-k)f_{m,n}(x+t,y+k)\,dt\,dk$$

$$=\sum_{y=0}^{L-1}\sum_{x=0}^{w-1}\int_y^{y+1}\int_x^{x+1}\bar{B}_{m,n}(\Theta-t,\Psi-k)f_{m,n}(t,k)\,dt\,dk$$

$$=\int_0^{L}\int_0^{w}\bar{B}_{m,n}(\Theta-t,\Psi-k)f_{m,n}(t,k)\,dt\,dk.$$

The remainder term is not $\sum_{y=0}^{L-1}\sum_{x=0}^{w-1}R_{m,n}$ but consists of the last five terms in the expression for $\sum_{y=0}^{L-1}\sum_{x=0}^{w-1}f(x+\Theta,y+\Psi)$.

If the function f, for which the sum is desired, is a polynomial in x and y the remainder terms are zero for large enough values of m and n.

Let Θ and Ψ both be equal to zero, then

$$\sum_{y=0}^{L-1}\sum_{x=0}^{w-1}f(x,y)=\int_0^{L}\int_0^{w}f(t,k)\,dt\,dk+\sum_{s=1}^{n}\sum_{v=1}^{m}\bar{B}_{v,s}f_{v-1,s-1}(x,y)\Big]_0^{w}\Big]_0^{L}$$

$$+\sum_{v=1}^{m}\bar{B}_{v,0}\int_0^{L}f_{v-1,0}(x,k)\,dk\Big]_0^{w}+\sum_{s=1}^{n}\bar{B}_{0,s}\int_0^{w}f_{0,s-1}(t,y)\,dt\Big]_0^{L}$$

$$-\sum_{v=1}^{m}\int_0^{L}\bar{B}_{v,n}(0,k)f_{v-1,n}(x,k)\,dk\Big]_0^{w}$$

$$-\sum_{s=1}^{n}\int_0^{w}\bar{B}_{m,s}(t,0)f_{m,s-1}(t,y)\,dt\Big]_0^{L}$$

$$-\int_0^{L}\bar{B}_{0,n}(0,k)\int_0^{w}f_{0,n}(t,k)\,dt\,dk$$

$$-\int_0^{w}\bar{B}_{m,0}(t,0)\int_0^{L}f_{m,0}(t,k)\,dk\,dt$$

$$-\int_0^{L}\int_0^{w}\bar{B}_{m,n}(t,k)f_{m,n}(t,k)\,dt\,dk.$$

This is the Euler-MacLaurin Summation Formula for two variables.

By using $\Delta_{\cdot t}B_{2r,0}(0,0) = B_{2r,0}(t,0) + B_{2r}$, and

$$\Delta_{\cdot t}\Delta_{\cdot k}B_{2r,2s}(0,0) = B_{2r,2s}(t,k) - B_{2r,2s}(0,k) - B_{2r,2s}(t,0) + B_{2r,2s},$$

the above becomes

$$\begin{aligned}
\sum_{y=0}^{L-1}\sum_{x=0}^{w-1} f(x,y) = {} & \int_0^L\int_0^w f(t,k)\,dt\,dk + \sum_{u=1}^{s-1}\sum_{v=1}^{r-1} \bar{B}_{2v,2u} f_{2v-1,2u-1}(x,y)\Big]_0^w\Big]_0^L \\
& + \sum_{u=1}^{s-1} \bar{B}_{1,2u} f_{0,2u-1}(x,y)\Big]_0^w\Big]_0^L + \sum_{v=1}^{r-1} \bar{B}_{2v,1} f_{2v-1,0}(x,y)\Big]_0^L\Big]_0^w \\
& + \bar{B}_{1,1} f_{0,0}(x,y)\Big]_0^w\Big]_0^L + \sum_{v=1}^{r-1} \bar{B}_{2v,0}\int_0^L f_{2v-1,0}(x,k)\,dk\Big]_0^w \\
& + \bar{B}_{1,0}\int_0^L f_{0,0}(x,k)\,dk\Big]_0^w + \sum_{u=1}^{s-1} \bar{B}_{0,2u}\int_0^w f_{0,2u-1}(t,y)\,dt\Big]_0^L \\
& + \bar{B}_{0,1}\int_0^w f_{0,0}(t,y)\,dt\Big]_0^L - \sum_{v=1}^{r-1}\int_0^L \bar{B}_{2v,0}\Delta_{\cdot k}\bar{B}_{0,2s} f_{2v-1,2s}(x,k)\,dk\Big]_0^w \\
& - \sum_{u=1}^{s-1}\int_0^w \bar{B}_{0,2u}\Delta_{\cdot t}\bar{B}_{2r,0} f_{2r,2u-1}(t,y)\,dt\Big]_0^L - \int_0^L \bar{B}_{1,0}\Delta_{\cdot k}\bar{B}_{0,2s} f_{0,2s}(x,k)\,dk\Big]_0^w \\
& - \int_0^w \bar{B}_{0,1}\Delta_{\cdot t}\bar{B}_{2r,0} f_{2r,0}(t,y)\,dt\Big]_0^L - \int_0^L \Delta_{\cdot k}\bar{B}_{0,2s}\Big[\int_0^w f_{0,2s}(t,k)\,dt\Big]dk \\
& - \int_0^w \Delta_{\cdot t}\bar{B}_{2r,0}\Big[\int_0^L f_{2r,0}(t,k)\,dk\Big]dt + \int_0^L\int_0^w \bar{B}_{2r,2s} f_{2r,2s}(t,k)\,dt\,dk.
\end{aligned}$$

By using the first theorem of the mean since $\Delta_{\cdot t}\bar{B}_{2r,0}$ is always positive, or does not change its sign,

$$\begin{aligned}
\int_0^w \bar{B}_{0,1}\Delta_{\cdot t}\bar{B}_{2r,0} f_{2r,0}(t,y)\,dt\Big]_0^L &= f_{2r,0}(g,y)\int_0^w \bar{B}_{0,1}\Delta_{\cdot t}\bar{B}_{2r,0}(0,0)\,dt\Big]_0^L \\
&= w\cdot f_{2r,0}(g,y)\int_0^1 \bar{B}_{0,1}\Delta_{\cdot t}\bar{B}_{2r,0}(0,0)\,dt \\
&= w\cdot \bar{B}_{2r,1} f_{2r,0}(g,y), \qquad (0 \leqq g \leqq w),
\end{aligned}$$

since $B_{2r,0}(t,0)$ is symmetrical to $(1/2, 0)$ and hence

$$\int_0^1 B_{2r,0}(t,y)\,dt\Big]_0^L = 0.$$

Using this idea for the integrals involving the Δ's, the above becomes

$$\sum_{y=0}^{L-1}\sum_{x=0}^{w-1} f(x,y) = \int_0^L\int_0^w f(t,k)\,dt\,dk + \sum_{u=1}^{s-1}\sum_{v=1}^{r-1} \bar{B}_{2v,2u} f_{2v-1,2u-1}(x,y)\Big]_0^w\Big]_0^L$$

$$+\sum_{v=1}^{r-1}\bar{B}_{2v,0}\int_0^L f_{2v-1,0}(x,k)\,dk\Big]_0^w+\bar{B}_{1,0}\int_0^L f(x,k)\,dk\Big]_0^w$$

$$+\sum_{u=1}^{s-1}\bar{B}_{0,2u}\int_0^w f_{0,2u-1}(t,y)\,dt\Big]_0^L+\sum_{u=1}^{s-1}\bar{B}_{1,2u}f_{0,2u-1}(x,y)\Big]_0^w\Big]_0^L$$

$$+\sum_{v=1}^{r-1}\bar{B}_{2v,1}f_{2v-1,0}(x,y)\Big]_0^L\Big]_0^w+\bar{B}_{1,1}f_{0,0}(x,y)\Big]_0^w\Big]_0^L$$

$$+\bar{B}_{0,1}\int_0^w f_{0,0}(t,y)\,dt\Big]_0^L-\sum_{v=1}^{r-1}\bar{B}_{2v,2s}L\cdot f_{2v-1,2s}(x,q_v)\Big]_0^w$$

$$-\sum_{u=1}^{s-1}\bar{B}_{2r,2u}w\cdot f_{2r,2u-1}(p_u,y)\Big]_0^L-\bar{B}_{1,2s}L\cdot f_{0,2s}(x,\bar{p}_1)\Big]_0^w$$

$$-\bar{B}_{2r,1}w\cdot f_{2r,0}(\bar{q}_2,y)\Big]_0^L-\bar{B}_{0,2s}L\cdot f_{0,2s}(\bar{q}_3,\bar{p}_3)$$

$$-\bar{B}_{2r,0}w\cdot f_{2r,0}(\bar{q}_4,\bar{p}_4)+L\cdot w\cdot\bar{B}_{2r,2s}f(\bar{q}_5,\bar{p}_6),$$

where

$$0\leqq q_i\leqq L,\qquad 0\leqq p_i\leqq w,$$
$$0\leqq\bar{q}_i\leqq L,\qquad 0\leqq\bar{p}_i\leqq w.$$

Let $$F_{i,j}(x,y)=f_{i-1,j-1}(x,y),$$

where the first subscript represents differentiation with respect to x and the second represents differentiation with respect to y. By using this relation, equation 6 becomes

$$\sum_{y=0}^{L-1}\sum_{x=0}^{w-1}f(x,y)=\sum_{s=1}^{n}\sum_{v=1}^{m}\bar{B}_{v,s}F_{v,s}(x,y)\Big]_0^L\Big]_0^w-\sum_{v=0}^{m}\int_0^L\bar{B}_{v,n}(0,k)F_{v,n+1}(x,k)\,dk\Big]_0^w$$

$$-\int_0^w\int_0^L\bar{B}_{m,n}(t,k)F_{m+1,n+1}(t,k)\,dt\Big]dk$$

$$-\sum_{s=0}^{n}\int_0^w\bar{B}_{m,s}(t,0)F_{m+1,s}(t,y)\,dt\Big]_0^L,$$

which makes the writing of the formula simplier.

If L and w approach infinity, sufficient conditions for the validity of this summation formula can be given. If $\sum_{y=0}^{\infty}\sum_{x=0}^{\infty}f(x+\Theta,y+\Psi)$ and $\int_0^\infty\int_0^\infty f(t,k)\,dt\,dk$ converge and if $f_{v-1,s-1}(w,L)$ approach zero as w and L approach infinity $(v=1,2,\cdots,m)$, $(s=1,2,\cdots,n)$, then

$$\sum_{y=0}^{\infty}\sum_{x=0}^{\infty}f(x+\Theta,y+\Psi)=\int_0^\infty\int_0^\infty f(t,k)\,dt\,dk+\sum_{s=1}^{n}\sum_{v=1}^{m}\bar{B}_{v,s}f_{v-1,s-1}(0,0)$$

$$+\sum_{v=1}^{m}\bar{B}_{v,0}\int_0^\infty f_{v-1,0}(0,k)\,dk+\sum_{s=1}^{n}\bar{B}_{0,s}\int_0^\infty f_{0,s-1}(t,0)\,dt$$

$$-\sum_{v=1}^{m}\int_0^\infty \bar{B}_{v,n}(0,k)f_{v-1,n}(0,k)\,dk$$

$$-\sum_{s=1}^{n}\int_0^\infty \bar{B}_{m,s}(t,0)f_{m,s-1}(t,0)\,dt$$

$$-\int_0^\infty \bar{B}_{0,n}(0,k)\int_0^\infty f_{0,n}(t,k)\,dt\,dk$$

$$-\int_0^\infty \bar{B}_{m,0}(t,0)\int_0^\infty f_{m,0}(t,k)\,dk\,dt$$

$$-\int_0^\infty\int_0^\infty \bar{B}_{m,n}(t,k)f_{m,n}(t,k)\,dt\,dk,$$

provided $\int_0^\infty f_{0,n}(t,k)\,dk$ and $\int_0^\infty f_{m,0}(t,k)\,dt$ exist.

UNIVERSITY OF MICHIGAN.

THE CELESTIAL SPHERE.

By F. Morley.

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1. *The Celestial Sphere.*

We are witnessing the arithmetizing of the physical universe as it now appears. The process is, on a grand scale, similar to the arithmetizing of space as it then appeared by Descartes and others.

The merger of Descartes for a plane is effected when we prove that there is a one-to-one correspondence between numbers x and the points of the plane. The numbers are of course the unrestricted numbers of algebra, usually called complex.

By an inversion from an outside point, we have the numbers x attached to the points of a sphere Ω, which we regard as lying in a euclidean space.

In terms of radius r, longitude θ, and colatitude ϕ, we may take

$$x = 2r \cot(\phi/2) \exp i\theta.$$

We now consider the interior of Ω only. The arc of a circle orthogonal to Ω is named by the end-points x and y on Ω, or collectively by a quadratic q whose zeros are x and y. The handling of arcs is then the handling of quadratics, $q_1, q_2, \cdots$.

2. *The Theory of Quadratics.*

Two ordered quadratics have a Jacobian j_{12} and a bilinear invariant q_{12}. They have then an absolute covariant j_{12}/q_{12}. The vanishing of q_{12} says that the two pairs of zeros are harmonic, whence the arcs intersect at right angles, or are *normal*. The Jacobian j_{12} is then the common normal of q_1 and q_2.

For 4 points of Ω we have three pairs of arcs, and three common normals. The theory of the quartic shows that these three quadratics q_i are themselves normal and meet at a point at right angles.

For any three quadratics we have as in Salmon's Higher Algebra the relation or *syzygy*

$$\begin{vmatrix} q_1 & q_2 & q_3 & 0 \\ q_{11} & q_{12} & q_{13} & q_1 \\ q_{21} & q_{22} & q_{23} & q_2 \\ q_{31} & q_{32} & q_{33} & q_3 \end{vmatrix} = 0.$$

This is familiar (for real numbers) as the equation of a conic, say

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

or
$$(a/fgh - 1/f^2)x^2 + \cdots + (x/f + y/g + z/h)^2 = 0.$$

That is, quadratics fall into sets of four, three determining a fourth, such that

$$\lambda_1 q_1^2 + \lambda_2 q_2^2 + \lambda_3 q_3^2 + \lambda_4 q_4^2 = 0,$$

for any value of the variable. Now $\lambda_1 q_1^2 + \lambda_2 q_2^2 = 0$ is four points giving a pair of arcs normal to the common normal of q_1 and q_2, j_{12}. But then also $\lambda_3 q_3^2 + \lambda_4 q_4^2 = 0$ is the same four points giving a different pair of arcs normal to j_{34}. By the theory of four points the arcs j_{12} and j_{34} are normal.

The set of four arcs is then such that the common normals j_{12}, j_{34} are normal, j_{23}, j_{14} are normal, and j_{31}, j_{24} are normal. This configuration of 10 arcs, each normal to three others, is in the restricted domain of real numbers the Descartes configuration.

There is between any seven quadratics

$$q_i \equiv \alpha_i x^2 + \beta_i x + \gamma_i$$

a quadric relation

$$\begin{vmatrix} \alpha_i^2 & \beta_i^2 & \gamma_i^2 & \beta_i\gamma_i & \gamma_i\alpha_i & \alpha_i\beta_i & q_i^2 \end{vmatrix} = 0,$$

and we ask the meaning of the 6-rowed condition on six quadratics

$$\begin{vmatrix} \alpha_i^2 & \beta_i^2 & \gamma_i^2 & \beta_i\gamma_i & \gamma_i\alpha_i & \alpha_i\beta_i \end{vmatrix} = 0.$$

In terms of the zeros it is the symmetric two-to-two or biquadratic correspondence, or involution. In the restricted domain it is familiar as Pascal's theorem when the six points α_i, β_i, γ_i are on a given conic, and the theory built on this theorem by Steiner, Plücker, Kirkman, Cayley, Salmon, Veronese, Cremona and Richmond is immediately available. When the six points are not given as on a conic (that is when each q_i is not a square) the theory in the restricted domain is that of Poncelet polygons.

In the present theory Pascal's theorem takes the form:

Let there be 6 arcs q_i in a biquadratic involution,

$$(\alpha x)^2(\alpha y)^2 + \Lambda(x - y)^2.$$

Ordering them say as 1 2 3 4 5 6, we have three pairs of common normals j_{12}, j_{45}. These three pairs have three common normals. These three have themselves a common normal. Cp. Klein, *Mathematische Annalen*, Vol. 22; *Works*, Vol. 1, p. 406.

If this happens for one ordering, it holds for all the 60 orderings. We have the 60 Pascal arcs.

It may be mentioned that the Kirkman arc for the above ordering is

$$j_{12}/q_{12} + j_{23}/q_{23} + j_{34}/q_{34} + j_{45}/q_{45} + j_{56}/q_{56} + j_{61}/q_{61}.$$

There is a cubic relation between any 11 quadratics namely

$$| \alpha_i^3 \cdot \cdot \cdot \alpha_i^2\beta_i \cdot \cdot \cdot \alpha_i\beta_i\gamma_i \cdot \cdot \cdot q_i^3 | = 0.$$

If q_{11} is arbitrary then we have 10 quadratics in a three-to-three or bicubic involution. And if also q_{10} is arbitrary, we have the fact that eight arcs determine a ninth, the relation being a mutual one.

3. *The Euclidean Case.*

There is now the interesting case when the universe Ω expands. We make the radius of the sphere Ω infinite. The arcs become lines of a euclidean space. The somewhat difficult subject of lines in a euclidean space, (see for instance Study, *Geometrie der Dynamen* and Blaschke, *Differential Geometrie,* Vol. 1, p. 264), so viewed is merely the humane theory of quadratics (under homographies). For example the configuration of 10 arcs becomes the Petersen-Morley configuration.

ON THE CONFIGURATION OF THE NINE BASE POINTS OF A PENCIL OF EQUIANHARMONIC CUBICS.

By J. YERUSHALMY.*

1. In a paper published in this Journal,† we proposed to show that every pencil of equianharmonic cubics is contained in a net of such cubics through 6 of the 9 base-points of the pencil, and that these 6 points form the vertices of two in-circumscribed triangles in a cubic ϕ which are three-fold perspective from the vertices of a third in-circumscribed triangle in the same cubic. However, the proof given there is incomplete. The purpose of this note is to complete the proof and to show that the 9 base points of a pencil of equianharmonic cubics divide into 3 triples i, j, k such that through any two triples i, j there passes a cubic ϕ_k in which they form the vertices of two in-circumscribed triangles three-fold perspective from the vertices of a third in-circumscribed triangle in ϕ_k. Thus through any two triples there passes a net of equianharmonic cubics.

2. The reasoning used in P was the following. The plane α of the pencil was mapped into a double plane π by means of a net of cubics on 7 of the 9 base-points. The branch-curve in π is a quartic, the equation of which was proved by Chisini ‡ to be

$$f_4 \equiv (\partial^2\phi_3/\partial x_3^2)\phi_3 - \tfrac{1}{2}(\partial\phi_3/\partial x_3)^2 = 0.$$

We considered the cubic surface

$$F_3 \equiv (\partial^2\phi_3/\partial x_3^2)x_4^2 + 2^{\frac{1}{2}}(\partial\phi_3/\partial x_3)x_4 + \phi_3 = 0,$$

which, when projected from the point $O \equiv (0, 0, 0, 1)$ on it upon the plane π $(x_4 = 0)$, gives $f_4 = 0$ as branch curve. The $(1-2)$ correspondence A between α and π and the $(1-2)$ correspondence B between F_3 and π defines a $(1-1)$ correspondence D between F_3 and α which sends the sections of F_3 by the planes on O into the net of cubics on the original 7 base-points. We showed that F_3 contains a net of equinaharmonic plane sections and that

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† "Construction of Pencils of Equianharmonic Cubics," *American Journal of Mathematics*, Vol. 53 (1931), pp. 319-332. This paper will be referred to in the sequel by the letter *P*.

‡ Chisini, "Sui fasci di cubiche a modulo costante," *Rendiconti del Circolo Matematico di Palermo*, Vol. 41 (1916).

it admits a cyclic homology Γ_3 of period 3 into itself. By a mapping C of F_3 into α the homology Γ_3 has as image a biquadratic transformation T cyclic of period 3 having the fundamental points of the direct and inverse transformation coincident, and from this follows the conclusion.

The weak point in the proof is that we assumed that the $(1-1)$-correspondence D between F_3 and α is necessarily a Clebsch mapping C of F_3 on α by means of which the plane sections of F_3 go into cubics on 6 of the 7 base-points of the original net. This would be so if and only if the double-tangent

$$d \equiv \partial^2 \phi_3 / \partial x_3^2 = 0,$$

which is the image of the point O on F_3 in the mapping B, would correspond in the mapping A to one of the 7 base-points in α. If, however, this double-tangent corresponds to a cubic degenerating into a conic on five of the base-points and a line on the other two base-points, then the correspondence D is such that either the point O goes by it into the line and the cubic f_3 cut out on F_3 by the tangent plane at O goes into the conic or vice versa. It is easily verified that in the first case D is a mapping M of F_3 unto α by means of which the plane sections of F_3 go into quartics on the 7 base-points

$$(1^2\, 2^2\, 3^1\, 4^1\, 5^1\, 6^1\, 7^1)_4$$

and in the second case D is a mapping N sending the plane sections of F_3 into quintics on the 7 base-points

$$(1^1\, 2^1\, 3^2\, 4^2\, 5^2\, 6^2\, 7^2)_5.$$

In both these mappings the net of cubics cut out on F_3 by the planes on O go into the original net of cubics on the 7 base-points $(1\,2\,3\,4\,5\,6\,7)_3$.

There remains hence the possibility of the existence of three distinct types (c), (m), (n) of pencils of equianharmonic cubics corresponding to the three different mappings. We prove, however, that there is only one type of pencil of equianharmonic cubic, by showing that the pencils of type (c), i. e. those considered in P give rise to all the three mappings, and every pencil of type (m) or (n) is also of type (c).

3. Take a pencil of equianharmonic cubics of type (c). The existence of such pencils was proved in P also independently of the cubic surface F_3 (*loc. cit.*, p. 331). Let $1, 2, \cdots, 9$ be the base-points of the pencil where $(1, 2, 3)$ $(4, 5, 6)$ are the vertices of two 3-fold perspective in-circumscribed triangles in a cubic ϕ_3, and are therefore the base-points of a periodic biquadratic transformation T

$$(1^2\, 2^2\, 3^2\, 4^1\, 5^1\, 6^1),$$

and 7, 8, 9 form a cycle for T. Consider an equianharmonic cubic ψ of the pencil. T leaves ψ invariant. Let the birational transformation of the cubic ψ into itself affected by T be

$$\gamma : u' = \epsilon u + b \qquad (u \text{ is the abelian parameter on } \psi, \; b \text{ a constant}, \; \epsilon^3 = 1).$$

From P, p. 329, we obtain

$$\begin{array}{ll} \text{a.} & \epsilon u_4 + b \equiv -(u_4 + u_5); \quad -\epsilon(u_1 + u_3) + b \equiv u_1; \quad \epsilon u_7 + b \equiv u_8 \\ (1) \quad \text{b.} & \epsilon u_5 + b \equiv -(u_5 + u_6); \quad -\epsilon(u_1 + u_2) + b \equiv u_2; \quad \epsilon u_8 + b \equiv u_9 \\ \text{c.} & \epsilon u_6 + b \equiv -(u_6 + u_4); \quad -\epsilon(u_2 + u_3) + b \equiv u_3; \quad \epsilon u_9 + b \equiv u_7, \end{array}$$

or

$$\begin{array}{ll} \text{a.} & \epsilon u_4 + b = -(u_4 + u_5) + \omega_1, \\ (2) \quad \text{b.} & \epsilon u_5 + b = -(u_5 + u_6) + \omega_2, \\ \text{c.} & \epsilon u_6 + b = -(u_6 + u_4) + \omega_3, \end{array} \qquad (\omega_i \text{ are linear combinations of the periods.})$$

from which $\omega_3 = -\epsilon^2\omega_2 - \epsilon\omega_1$. Substituting for ω_3 in 2c. and adding a, b, c, we obtain

$$u_4 + u_5 + u_6 = (\epsilon - 1)b + [(1 - \epsilon^2)\omega_2 - \epsilon(1 - \epsilon^2)\omega_1]/(1 - \epsilon^2),$$

or

$$(3) \qquad u_4 + u_5 + u_6 \equiv (\epsilon - 1)b.$$

Similarly it can be shown that

$$\begin{array}{llll} & \text{(a)} & u_1 + u_2 + u_3 \equiv \epsilon(\epsilon - 1)b & \text{(b)} \quad u_7 + u_8 + u_9 \equiv (\epsilon + 2)b \\ (4) & \text{(c)} & u_5 \equiv \epsilon^2 u_4 - b & \text{(d)} \quad u_6 \equiv \epsilon u_4 + \epsilon b \\ & \text{(e)} & u_2 \equiv \epsilon^2 u_1 - \epsilon b & \text{(f)} \quad u_3 \equiv \epsilon u_1 + \epsilon^2 b. \end{array}$$

Consider now a biquadratic transformation S with fundamental points at $(1^1\,2^1\,3^1\,7^2\,8^2\,9^2)$. The fundamental points of the inverse transformation S^{-1} will generally be some other 6 points $(1_0^2\,2_0^2\,3_0^2\,7_0^1\,8_0^1\,9_0^1)$. S sends the cubics on 1, 2, 3, 7, 8, 9 into the cubics on $1_0, 2_0, 3_0, 7_0, 8_0, 9_0$. It sends ψ into some cubic ψ_0.

The homoloidal net Σ' on 1, 2, 3, 7, 8, 9 cuts out on ψ a g_3^2. But since by 4a and 4b

$$u_1 + u_2 + u_3 \equiv \epsilon(\epsilon - 1)b,$$
$$2(u_7 + u_8 + u_9) \equiv 2(\epsilon + 2)b,$$

and hence

$$u_1 + u_2 + u_3 + 2(u_7 + u_8 + u_9) \equiv 3b,$$

the sum of the abelian parameter at a set of the g_3^2 is equivalent to $-3b$. Also if we transform any set of 3 points a, b, c on a line by a transformation $\bar{\gamma} : u' = \epsilon^2 u - b$ we obtain

$$u'_a = \epsilon^2 u_a - b\,; \quad u'_b = \epsilon^2 u_b - b\,; \quad u'_c = \epsilon^2 u_c - b,$$

from which

$$u'_a + u'_b + u'_c \equiv \epsilon^2 (u_a + u_b + u_c) - 3b.$$

But since

$$u_a + u_b + u_c \equiv 0$$

$$\therefore \quad u'_a + u'_b + u'_c \equiv -3b.$$

Hence $\bar{\gamma}$ transforms the g_3^2 cut out on ψ by the lines of the plane into the g_3^2 cut out on it by the quartics of Σ'. Therefore, the product $\bar{\gamma}S$ is a birational transformation of ψ into ψ_0 sending the g_3^2 cut out on ψ by the lines of the plane into the g_3^2 cut out on ψ_0 by the lines of the plane. Hence $\bar{\gamma}S$ is involved by a collineation λ between ψ and ψ_0. The collineation sends the points $1, 2, 3, 7, 8, 9$ into the points $1_0\, 2_0\, 3_0\, 7_0\, 8_0\, 9_0$. To prove this take for example the point 1. $\bar{\gamma}$ sends it into the point $\epsilon^2 u_1 - b$, which by 4b and 4f, is the sixth point of intersection of ψ with the conic $(1, 3, 7, 8, 9)$ and this point goes by S into one of the points $1_0, 2_0, 3_0$, say 1_0. In the same way it is easily seen that λ sends 2 into 2_0 etc. Multiplying λ by S^{-1} we obtain the birational transformation $\bar{\gamma}$ which is evidently a biquadratic transformation sending ψ into itself and having the fundamental points of the direct and inverse transformation coincident which is sufficient to prove the existence of a net of equianharmonic cubics on $1, 2, 3, 7, 8, 9$. A similar reasoning shows that there exists a net of equianharmonic cubics also on $4, 5, 6, 7, 8, 9$. There are therefore defined two other cubics ϕ_2 and ϕ_1 such that each one of them has two triples as vertices of two in-circumscribed triangles 3-fold perspective from the vertices of a third in-circumscribed triangle in it. Each of the three cubics ϕ_i passes through the cusps of the 6 cuspidal cubics of the pencil (Pp. 332).

There obviously can be no more than 3 nets of equianharmonic cubics on these points. For if there were a fourth it would define another cubic ϕ which would have with one of the 3 cubics ϕ_i more than 9 points in common since it would also have to pass through the cusps and have two inscribed triangles with vertices among the 9 base-points.

4. The division of the 9 base-points into 3 triples shows that a pencil of type (c) gives rise also to the mappings M and N. For we can select 7 of the 9 base-points in 2 essentially distinct ways: either they contain 2 triples and one point of the third triple, or they consist of one riple and of two points of each of the remaining 2 triples. If we map α on π by means of cubics on 7 points containing 2 triples, then the double tangent d of $f_4 = 0$ corresponding to O on F_3 will be the image of the seventh base-points and the

correspondence D between F_3 and α is a Clebsch mapping C. If, however, the seven points contain only 1 triple, then the double tangent d is necessarily the image of a cubic composed of a conic on 5 of the 7 points and a line on the other 2 points. Then, if in the mapping of F_3 on α, we let O correspond to the line and the cubic f_3 — to the conic, we obtain the mapping M, and if we let O correspond to the conic and f_3 to the line, we obtain the mapping N.

5. To complete the proof given in P, we must show now that every pencil of type (m) is also of type (c). Take such a pencil. Let $1, 2, \cdots, 9$ be its base-points. In the mapping M between F_3 and α the plane-sections of F_3 go into the quartics

$$(1^2\,2^2\,3^1\,4^1\,5^1\,6^1\,7^1)_4$$

and the line (12) is the image of the point O. It can be shown that the homology Γ_3 of F_3 is mapped by M into a Cremona transformation T_7 of order 7 with 8 fundamental points, 7 of which coincide for T_7 and T_7^{-1}. The homoloidal nets are the following:

$$\Sigma' : (1^3\,2^1\,3^1\,4^2\,5^2\,6^2\,7^3\,9^1)_7,$$
$$\Sigma : (1^4\,2^3\,3^3\,4^2\,5^2\,6^2\,7^1\,8^1)_7.$$

Knowing the configuration of the 27 lines on F_3 (they distribute in 9 triples, each triple is in a plane and the three lines concur at a point on the cubic ϕ), we can deduce the fundamental curves of T_7:

(5)

(a) $1 \to$	$(1^2\,2^1\,3^1\,4^1\,5^1\,6^1\,8^1)_3$	$\to$	$(1^2\,2^2\,3^1\,4^1\,5^1\,6^1\,7^2\,9^1)_4$	$\to 1$
(b) $2 \to$	$(1^2\,2^2\,3^2\,4^1\,5^1\,6^1\,7^1\,8^1)_4$	$\to$	$(1^1\,2^2\,4^1\,5^1\,6^1\,7^1\,9^1)_3$	$\to 2$
(c) $3 \to$	$(1\,3)_1$	$\to$	$(1^1\,2^2\,3^1\,4^1\,5^1\,6^1\,7^1)_3$	$\to 3$
(d) $4 \to$	$(1\,2\,4\,3\,5)_2$	$\to$	$(1\,2\,4\,6\,7)_2$	$\to 4$
(e) $5 \to$	$(1\,2\,5\,3\,6)_2$	$\to$	$(1\,2\,5\,4\,7)_2$	$\to 5$
(f) $6 \to$	$(1\,2\,6\,3\,4)_2$	$\to$	$(1\,2\,6\,5\,7)_2$	$\to 6$
(g) $7 \to$	$(1^2\,2^1\,3^1\,4^1\,5^1\,6^1\,7^1)_3$	$\to$	$(2\,7)_1$	$\to 7$
(h) $8 \to$	9	$\to$	$(1\,2)_1$	$\to 8$
(j) $9 \to$	$(1\,2)_1$	$\to$	8	$\to 9.$

The transformation T_7 leaves invariant each cubic of the original pencil. Take a cubic ψ of this pencil and let the birational transformation of ψ into itself involved by T_7 be

$$\gamma : u' = \epsilon u + b.$$

From 5c, g, h, j, we obtain

$$(6)\qquad \begin{aligned} \epsilon u_3 + b &\equiv -(u_1 + u_3), & -\epsilon(u_2 + u_7) + b &\equiv u_7, \\ \epsilon u_9 + b &\equiv -(u_1 + u_2), & -\epsilon(u_1 + u_2) + b &\equiv u_8, \end{aligned}$$

from which

$$(7)\qquad \begin{aligned} &\text{a. } u_3 \equiv \epsilon u_1 + \epsilon b & &\text{b. } u_7 \equiv \epsilon^2 u_2 - \epsilon b \\ &\text{c. } u_8 \equiv -\epsilon u_1 - \epsilon u_2 + b & &\text{d. } u_9 \equiv -\epsilon^2 u_1 - \epsilon^2 u_2 - \epsilon^2 b. \end{aligned}$$

Consider now a biquadratic transformation T_4 with fundamental points at $(1^1\,7^1\,8^1\,2^2\,3^2\,9^2)$. The fundamental points of T_4^{-1} will be

$$(1_0^2\,7_0^2\,8_0^2\,2_0^1\,3_0^1\,9_0^1).$$

The transformation T_4 sends ψ into ψ_0. The quartics Φ' : $(1^1\,7^1\,8^1\,2^2\,3^2\,9^2)_4$ cut out on ψ a g_3^2. But since by (7)

$$\begin{aligned} u_1 + u_7 + u_8 &\equiv (1-\epsilon)u_1 + \epsilon(\epsilon-1)u_2 + (1-\epsilon)b, \\ 2(u_2 + u_3 + u_9) &\equiv 2\epsilon(1-\epsilon)u_1 + 2(1-\epsilon^2)u_2 + 2\epsilon(1-\epsilon)b, \end{aligned}$$

and hence

$$u_1 + u_7 + u_8 + 2(u_2 + u_3 + u_9) \equiv -3\epsilon^2 u_1 + 3u_2 - 3\epsilon^2 b,$$

it follows that the sum of the abelian parameter at a set of the g_3^2 is equal to $3\epsilon^2 u_1 - 3u_2 + 3\epsilon^2 b$. It is easily seen that the birational transformation

$$\bar{\gamma} : u' = \epsilon^2 u + \epsilon^2 u_1 - u_2 + \epsilon^2 b$$

of ψ into itself sends the g_3^2 of the line sections into the g_3^2 cut out by Φ'. Therefore $\bar{\gamma}T_4$ is involved by a collineation λ sending ψ into ψ_0. λ sends the points $1, 7, 8, 2, 3, 9$ into the points $1_0, 7_0, 8_0, 2_0, 3_0, 9_0$. For, the point 1 for example, goes by $\bar{\gamma}$ into the point $2\epsilon^2 u_1 - u_2 + \epsilon^2 b$ which by (7) is the sixth intersection of ψ with the conic $(1\,7\,2\,3\,9)$ which goes by T_4 into one of the points $1_0, 7_0, 8_0$, say 1_0, and in the same way 2 goes into 2_0 etc. The product λT_4^{-1} gives $\bar{\gamma}$ which is evidently a biquadratic transformation having the fundamental points of the direct and inverse transformation coincident, and hence there is a net of equianharmonic cubics on $(1\,7\,8\,2\,3\,9)$ from which follows that every pencil of type (m) is also of type (c).

Since in the mapping M the net of equianharmonic sections of F_3 go into equianharmonic quartics we conclude also that on 7 of the 9 base-points of a pencil of equianharmonic cubics containing only one triple there is a net of equianharmonic elliptic quartics.

THE SYMMETRIC (n, n)-CORRESPONDENCE AND SOME GEOMETRIC APPLICATIONS.

By Arnold Emch.

I. Introduction.

Two parameters λ and μ shall respectively determine two geometric entities of the same kind uniquely. For example, the values of either λ or μ may determine the tangents of a conic, or the planes of a pencil, or the osculating planes of a space cubic, or the quadrics of a pencil, etc. We then can set up an algebraic relation between λ and μ as the variables and study the properties of this relation. To these correspond in the geometric association of λ and μ definite geometric properties which by this method are often obtained in a surprisingly simple manner when compared with the arduous synthetic labors necessary to obtain the same result. This will be apparent in the applications to the Schur sextic with ∞^1 inscribed pentahedrons. This interesting space curve has received very little attention in the literature and deserves detailed study. The same is true of the Lüroth quartic with its ∞^1 inscribed pentalaterals and its connection with the Schur sextic and the attached octavic ruled surface.

II. The Symmetric (n, n)-Correspondence.

Although there is nothing essentially new in the algebraic exposition of this correspondence,* I shall give here a short account which will be needed subsequently.

Let $\lambda + \mu = x$, $\lambda\mu = y$, then a polynomial of degree n in x and y

$$R(x, y) = 0 \tag{1}$$

represents such a correspondence. It depends evidently on $\frac{1}{2}n(n+3)$ constants and represents a general n-ic in the (x, y)-plane and is therefore of genus $\frac{1}{2}(n-1)(n-2)$. In the (λ, μ)-plane it is a curve of order $2n$ with the infinite points of the λ- and μ-axis as n-fold points, whose genus is $\frac{1}{2}(2n-1)(2n-2) - n(n-1) = (n-1)^2$. Geometrically, the correspondence between λ and μ may be obtained as follows: Associate λ and μ

* See A. B. Coble, "Multiple Binary Forms with Closure Property," *American Journal of Mathematics*, Vol. 43 (1921), pp. 1-19.

as parameters with the tangents of a fixed conic K. In the plane of K choose a general n-ic C_n as given by (1). From every point of C_n draw the two tangents to K and denote their parameters by λ and μ; then the algebraic relation between these parameters is precisely of the symmetric (n, n)-type. From this follows

THEOREM 1. *The pairs* $(\lambda, \mu) \equiv (\mu, \lambda)$ *of the symmetric* (n, n)*-correspondence are in birational correspondence with the points of a* C_n *of genus* $\frac{1}{2}(n-1)(n-2)$. There exists a $(1, 2)$-correspondence between C_n and C_{2n}. To every point of C_n correspond two points of C_{2n} which are symmetric with respect to the line $\lambda - \mu = 0$ in the (λ, μ)-plane.

In case of a "complete symmetry," as I define it, it must be true that for a generic value of $\lambda = \lambda_1$, we obtain n values for μ, $\mu = \lambda_2, \lambda_3, \cdots, \lambda_{n+1}$, such that every couple $(\lambda = \lambda_i,\ \mu = \lambda_k)$; $(i, k = 1, 2, \cdots, n+1;\ i \neq k)$, is also a pair of corresponding values (λ, μ), or a solution of (1). Geometrically these values determine two pencils of parallel lines

$$\lambda = \lambda_1, \lambda_2, \cdots, \lambda_{n+1}, \qquad \mu = \lambda_1, \lambda_2, \cdots, \lambda_{n+1}$$

which intersect in $(n+1)^2$ points; or, omitting the points on $\lambda - \mu = 0$, in $n(n+1)$ points of the curve C_{2n}. These form $\frac{1}{2}n(n+1)$ symmetric pairs (λ, μ); (μ, λ) with respect to the line $\lambda - \mu = 0$. Each of these pairs determines one of the $\frac{1}{2}n(n+3)$ constants of (1). Thus there are $\frac{1}{2}n(n+3) - \frac{1}{2}n(n+1) = n$ left to be determined, which can be done by the series of new couples $(\lambda, \mu) = (\lambda_{n+1}, \lambda_{n+2})$, $(\lambda_{n+1}, \lambda_{n+3}), \cdots, (\lambda_{n+1}, \lambda_{2n+1})$. Since the C_{2n} passes through the above $n(n+1)$ points it must necessarily have the form, after compounding it with $\lambda - \mu = 0$, and the line at infinity,

$$(2) \qquad C^*_{2n} = \prod_{i=1}^{n+1} (\lambda - \lambda_i) P(\lambda, \mu) + \prod_{i=1}^{n+1} (\mu - \lambda_i) Q(\lambda, \mu) = 0,$$

in which P and Q are of degree $n+1$. But since C_{2n} has the infinite points of the λ- and μ-axis as n-fold points, P and Q must be polynomials of degree $n+1$ in λ and μ alone, respectively, so that each can be resolved into $n+1$ linear factors

$$P = b \prod_{i=n+2}^{2n+2} (\mu - \lambda_i), \qquad Q = a \prod_{i=n+2}^{2n+2} (\lambda - \lambda_i).$$

As (2) must vanish identically, for $\lambda = \mu$, there must be $b = -a$, so that the relation is finally

$$(3) \qquad \prod_{i=1}^{n+1} (\lambda - \lambda_i) \prod_{i=n+2}^{2n+2} (\mu - \lambda_i) - \prod_{i=1}^{n+1} (\mu - \lambda_i) \prod_{i=n+1}^{2n+2} (\lambda - \lambda_i) = 0.$$

After division by $\lambda - \mu$ and cancellation of $\lambda^{n+1}\mu^{n+1}$ this becomes the completely symmetric (n, n)-correspondence. Choosing

$$\prod_{i=1}^{n+1} (\mu - \lambda_i) / \prod_{i=n+2}^{2n+2} (\mu - \lambda_i) = k$$

as a parameter, (3) appears as an involution of order $n+1$ on the line λ, so that *"complete symmetry" is equivalent with the involutorial property.* (3) may be considered as the product of the two projective involutorial pencils, with t as a parameter:

$$\prod_{i=1}^{n+1} (\lambda - \lambda_i) + t \prod_{i=n+2}^{2n+2} (\lambda - \lambda_i) = 0,$$

$$\prod_{i=1}^{n+1} (\mu - \lambda_i) + t \prod_{i=n+2}^{2n+2} (\mu - \lambda_i) = 0.$$

For every value of t we obtained two pencils of $n+1$ parallel lines each, which intersect in $n(n+1)$ points of the curve C_{2n}. Thus, there are ∞^1 such sets. The result may be stated as

THEOREM 2. *The symmetrical involutorial (n, n)-correspondence between λ and μ is completely determined by two involutorial sets of $\frac{1}{2}n(n+1)$ couples (λ, μ) each. If the correspondence contains one involutorial set, then it contains ∞^1 such sets. One involutorial set and n other couples $(\lambda_{n+2}, \lambda_i)$ $(i = n+3, \cdots, 2n+2)$, determine the correspondence completely.*

III. THE PARAMETERS λ, μ ASSOCIATED WITH THE TANGENTS OF A CLASS-CONIC.

1. *The involutorial (n, n)-correspondence.* Again let

$$R(x, y) = 0, \qquad x = \lambda + \mu, \qquad y = \lambda\mu, \tag{4}$$

represent such a correspondence, and λ the parameter which for every value determines a tangent

$$a\lambda^2 + b\lambda + c = 0 \tag{5}$$

of a class-conic K, in which a, b, c are linear forms of a plane $(x) = (x_1, x_2, x_3)$. Such a tangent is likewise

$$a\mu^2 + b\mu + c = 0. \tag{6}$$

For every couple (λ, μ) satisfying (4), (5) and (6) represent two tangents of K which intersect in a point P of a certain curve. To find the equation of this curve, λ and μ must be eliminated between (4), (5), (6). For this

purpose subtract (5) from (6) and divide the result by $\lambda - \mu$. This gives $ax + b = 0$. Again multiply (5) and (6) by μ^2 and λ^2 respectively, subtract and divide by $\lambda - \mu$, which gives $by + cx = 0$. Solving the two resulting equations, we get $x = -b/a$, $y = c/a$, which on substitution in (4) give an equation of degree n in x_1, x_2, x_3, representing a curve C_n of order n. As there are two tangents (λ), (μ) from every point of C_n to K, the n-ic is evidently in birational relation with the pairs $(\lambda, \mu) \equiv (\mu, \lambda)$ or with the n-ic $R(x, y)$ in the (x, y)-plane. The genus of C_n is therefore $\frac{1}{2}(n-1)(n-2)$. Moreover every tangent (λ) determines an involutorial $(n+1)$-lateral which is inscribed to the C_n, i. e., whose $\frac{1}{2}n(n+1)$ points of intersection lie on C_n. Hence

THEOREM 3. *The involutorial (n, n)-correspondence determines a curve of order n of genus $\frac{1}{2}(n-1)(n-2)$ with ∞^1 completely inscribed $(n+1)$-laterals, whose sides touch a fixed conic.* For $n = 2, 3, 4$, we have the following well-known results:

2. *Triangles inscribed and circumscribed to two conics respectively.* For $n = 2$, the involutorial correspondence determines a conic C_2 to which are inscribed ∞^1 triangles whose sides touch a given conic K.

3. *Complete quadrilaterals inscribed in a cubic.* The case $n = 3$ leads to an elliptic cubic C_3 with ∞^1 quadrilaterals circumscribed to a conic with their vertices on the fixed cubic. The result may be stated differently as

THEOREM 4. *Two sextuples of vertices formed by each of two distinct quadrilaterals circumscribed to a fixed conic determine a plane cubic uniquely. There are ∞^1 quadrilaterals inscribed to the cubic and circumscribed to the conic.*

4. *The Lüroth quartic.* When $n = 4$, we get a quartic of genus 3 with ∞^1 completely inscribed pentalaterals whose sides envelope a fixed conic. This quartic was found by Lüroth * in connection with the study of the Clebsch quartic,† which is characterized by the existence of an apolar conic, or its representability by the sum of the squares of five linear forms. These represent in every case the five sides of a pentalateral whose ten vertices generate the Lüroth quartic.

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* "Einige Eigenschaften einer gewissen Gattung von Curven vierter Ordnung," *Mathematische Annalen*, Vol. 1 (1869), pp. 37-53.

† "Ueber Curven vierter Ordnung," *Crelle's Journal für reine und angewandte Mathematik*, Vol. 59 (1861), pp. 125-145.

Propositions of this and a similar sort have also been investigated in a different connection by W. F. Meyer.*

IV. The Involutorial (n, n)-Correspondence on a Developable Cubic.

1. *General case.* Assume a fixed developable cubic surface D with the likewise rational cubic edge of regression Γ and its tangents t. D is of order 4; i. e., through every generic line in space there pass four tangents t of Γ, and every t is the intersection of two infinitely close planes of D. This surface may be represented in the form

$$a\lambda^3 + b\lambda^2 + c\lambda + d = 0, \tag{7}$$

in which a, b, c, d are linear forms in $S_3(x_1, x_2, x_3, x_4)$. For every value of λ, (7) is one of the planes (7) of D. Likewise

$$a\mu^3 + b\mu^2 + c\mu + d = 0 \tag{8}$$

is such a plane. The parameters λ and μ are now subjected to the involutorial relation expressed by (1), respectively (3). Every couple (λ, μ) satisfying this relation determines two planes of D which intersect in a generatrix g of a certain ruled surface G, whose equation is again obtained by eliminating λ and μ between (1), (7) and (8). Setting, as before, $\lambda + \mu = x$, $\lambda\mu = y$, then by simple manipulations of (7) and (8) the equations result:

$$a(x^2 - y) + bx + c = 0, \tag{9}$$

$$by^2 + cxy + d(x^2 - y) = 0, \tag{10}$$

The elimination gives first an equation of degree $4n = n \cdot 2 + n \cdot 2$. Now (9) and (10) are satisfied by $x = -c/b$, $y = x^2 = (c/b)^2$, which solution does not satisfy (1). As a consequence the result of the elimination is reduced by $2n$, so that for the order of the ruled surface G proper, the number $2n$ is obtained. A generic plane (λ_1) determines an involutorial group of $n + 1$ planes (λ_1), $(\lambda_2), \cdots, (\lambda_{n+1})$ on D, so that on each of these, the remaining n cut out on n-lateral with $\frac{1}{2} \cdot n(n-1)$ points of intersection. In each of these planes there are no other points in which two planes of the involutorial set are concurrent. The points of the n-lateral in (λ_i), as λ_1 varies, describe therefore a space curve C_m of order $m = \frac{1}{2}n(n-1)$. Any two planes of the $(n+1)$-edron intersect in a generatrix g of G, and g is cut by $n-1$ planes of the involutorial group and becomes thus an $(n-1)$-fold secant of the space curve. Moreover through

* *Apolarität und rationale Curven* (1883).

every point of C_m there are three $(n-1)$-fold secants, so that C_m is a triple curve of G. A plane (λ) of D cuts G in n lines g and a residual C_n to which the n-lateral (g) is inscribed. A generic plane p cuts G in a C_{2n} which at the $m = \frac{1}{2}n(n-1)$ intersections of C_m with p has triple points. The genus of C_{2n} is therefore

$$\pi = \tfrac{1}{2}(2n-1)(2n-2) - 3 \cdot \tfrac{1}{2}n(n-1) = \tfrac{1}{2}(n-1)(n-2).$$

Both C_{2n} and C_n are on the same ruled surface G and hence birationally related, so that C_n has also the genus π, and is therefore without singularities. The genus σ of C_m is easily found from the projection C_m of C_m from one of its generic points upon a generic plane. C_m has the order $\frac{1}{2}n(n-1)-1$ and has three $(n-2)$-fold points arising from the three $(n-1)$-fold secants of C_m through the center of projection. Thus

$$\sigma = \tfrac{1}{2}\,[\,\tfrac{1}{2}n(n-1)-2\,]\,[\,\tfrac{1}{2}n(n-1)-3\,] - 3 \cdot \tfrac{1}{2}(n-2)(n-3),$$

which reduces to

$$\sigma = 1/8(n^4 - 2n^3 - 21n^2 + 70n - 48).$$

A plane $(\lambda) = (\lambda_1)$ of D and an infinitely close plane (λ'_1) intersect in a tangent t of Γ, the curve of regression of D. The remaining planes of the involutorial group determined by (λ_1), cut (λ_1) in the n generatrices $g_2, g_3, \cdots, g_{n+1}$. These cut t in n points $P_2, P_3, \cdots, P_{n+1}$ which are on G as well as on D. Likewise to (λ'_1) correspond n planes which cut (λ'_1) in n generatrices $g'_2, g'_3, \cdots, g'_{n+1}$ which are infinitely close to $g_2, g_3, \cdots, g_{n+1}$, respectively, and which cut t in n points $P'_2, P'_3, \cdots, P'_{n+1}$, in the same order infinitely close to $P_2, P_3, \cdots, P_{n+1}$. Hence t touches G in n points $P_2, P_3, \cdots, P_{n+1}$. The plane (λ_1) cuts G in $g_2, g_3, \cdots, g_{n+1}$, and a residual C_n. This curve cuts t in n points which are precisely the points $P_2, P_3, \cdots, P_{n+1}$, so that these are the points of tangency of (λ_1) with G. From this follows that D as a surface of order 4 touches G along a curve of order $4n$, which counted twice is the complete intersection of D and G. In conclusion we have

THEOREM 5. *The $\frac{1}{2}n(n+1)$ lines of intersection g of the planes of a group of $n+1$ planes corresponding to the groups of an involutorial (n,n)-correspondence on a cubic surface D of class 3 generate a ruled surface G of order $2n$ with a space curve C_m of order $m = \frac{1}{2}n(n-1)$ and genus $\frac{1}{2}(n^4 - 2n^3 - 21n^2 + 70n - 48)$ as a triple curve. Every generatrix of G is an $(n-1)$-fold secant of C_m. Every plane of D cuts G in n generatrices and a residual C_n. The residual n intersections of these generatrices with C_n*

are collinear on a line t which is a tangent of the cubic of regression of D. The latter surface is a complete contact surface of G and touches G along a curve of order $4n$.

2. *A sextic surface with a space cubic as a triple curve.* This surface G is obtained for $n = 3$ and was first investigated by E. Weyr * and also by the author † as an example of a more general class of surfaces.

The preceding theorem 5 for this case may be stated in a form found by A. Hurwitz ‡ in a remarkable study on problems of closure:

THEOREM 6. *The eight faces of any two tetrahedrons inscribed in a space cubic are osculating planes of another space cubic. Both curves are related suchwise that there are ∞^1 such tetrahedrons with the same property. Every point of the first curve is a vertex of just one of these tetrahedrons.*

The two curves are $C_m = C_3$ of genus $\sigma = 0$, and Γ. Each face of a tetrahedron inscribed to C_3 and circumscribed to Γ cuts G in three generatrices g and a residual elliptic cubic $C_n = C_3$. The g's cut C_3 in three residual points which are collinear on a tangent t of Γ. D touches G along a curve of order 12.

3. *The Schur sextic.* In 1881 F. Schur published a paper on curves and surfaces generated by collinear forms (Grundgebilde) in which, among other interesting results, he found a space sextic with remarkable properties. This is the curve $C_m = C_6$ of genus $\sigma = 3$, when $n = 4$ in the involution on the developable surface D of class 3. The five planes of an involutorial group form a pentahedron completely inscribed in the C_6; i. e., its 10 edges are trisecants and through each of the 10 vertices pass three trisecants. On account of the importance of this curve it is perhaps not superfluous to restate the general results for this particular case in the form of

THEOREM 7. *Among the space sextics C_6 of genus 3 there exists a class in which every curve admits of ∞^1 inscribed pentahedrons. The edges of these pentahedrons generate for every curve C_6 a ruled surface G of order 8 with C_6 as a triple curve. If a C_6 has one inscribed pentalateral, then it has an infinite number. The faces of the pentahedrons envelope a developable*

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* "Ueber Flächen sechsten Grades mit einer dreifachen cubischen Curve," *Wiener Sitzungsberichte*, Vol. 85 (1882), pp. 513-525.

† "Ueber eine besondere Klasse von algebraischen Flächen," *Commentarii Mathematici Helvetici*, Vol. 2 (1930), pp. 99-115.

‡ "Ueber unendlich-vieldeutige geometrische Aufgaben, insbesondere über Schliessungsprobleme," *Mathematische Annalen*, Vol. 15 (1879), pp. 8-15.

surface D of class 3 and order 4 with a cubic curve of regression Γ. D is a contact surface of G. Every face of an inscribed pentahedron cuts G in a quadrilateral (g) and a residual quartic C_4. The sides of (g) cut C_4 in four residual points, collinear on a tangent t of Γ, so that (g) and t form an inscribed pentalateral of C_4. These residual curves are therefore Lüroth quartics. The manifold of Schur sextics is ∞^{20}.

The concluding statement may easily be verified. Two arbitrary planes (λ_1) and (λ'_1) of D cut every C_6 of the class. With (λ_1) we can associate four other planes of D to form a pentahedron Δ; likewise with (λ'_1) another distinct set of four planes of D forms a second pentahedron Δ'. The two Δ and Δ' determine a Schur sextic uniquely and may be chosen in $\infty^4 \cdot \infty^4 = \infty^8$ ways. On the other hand D is determined by six generic planes (λ). But on a given D six planes may be chosen in ∞^6 ways. The manifold of D's is therefore $\infty^{18} \cdot \infty^6 = \infty^{12}$, and consequently the manifold of Schur sextics $\infty^8 \cdot \infty^{12} = \infty^{20}$. It will be remembered that it is ∞^{24} for the general sextic of genus 3.

4. *The Schur sextic as a locus of vertices of cones.* In Schur's paper can also be found the proof that the sextic discussed above is also a "Kegelspitzenkurve," i. e., the locus of vertices of cones in a certain net of quadrics, or the Jacobian curve of this net. Now the manifold of such Jacobian curves is ∞^{21}. Putting the condition on such a curve that it shall lie on a six-point of a quadrilateral diminishes the manifold by 1, so that the manifold of these Jacobian curves is ∞^{20}, the same as that of the Schur sextics.

The fact that the Schur sextic is the Jacobian curve of a net of quadrics may easily be verified by the following proof which may be given in outline. Take any of the inscribed pentahedrons of a C_6. It contains 5 tetrahedrons. On each face of such a tetrahedron there lies a quadrilateral of trisecants with a diagonal triangle. Thus on each of the tetrahedrons there are 12 points which lie on a quadric Q, and every inscribed pentahedron determines 5 quadrics which may be shown to be linearly independent and form a linear four dimensional system $\sum_{i=1}^{5} a_i Q_i = 0$, with respect to which the pairs of vertices of every quadrilateral of trisecants are apolar. Two such systems of quadrics based on two distinct inscribed pentalaterals have a net of quadrics in common, whose fundamental curve in the involutorial cubic transformation determined by the net is identical with the Schur sextic. Thus

THEOREM 8. *The Jacobian curve of a net of quadrics is a Schur sextic when it is on the six points of a plane quadrilateral.*

A HYPERSURFACE OF ORDER 2^{r-1} IN r-SPACE.

By B. C. Wong.

1. The equation

$$x_1^{\frac{1}{2}} \pm x_2^{\frac{1}{2}} \pm \cdots \pm x^{\frac{1}{2}}_{r+1} = 0, \tag{1}$$

when rationalized, is of degree 2^{r-1} and therefore represents a hypersurface $V^{2^{r-1}}_{r-1}$ of order 2^{r-1} in r-space. If $r = 2$ and 3, we have a conic and a Steiner's quartic surface respectively. In this paper we propose to study the hypersurface for $r > 3$. We shall first obtain the different multiple varieties on the hypersurface, then give a few general remarks concerning its 3-space and plane sections. We shall, finally, consider its representation upon a hyperplane.

2. For the purpose of obtaining the different multiple varieties on $V^{2^{r-1}}_{r-1}$, we find it convenient to group all the terms in the left-hand member of (1) into ν groups. Letting the k-th group contain t_k terms, we write the equation in the form

$$\sum_1^{t_1} x_{i_1}^{\frac{1}{2}} \pm \sum_1^{t_2} x_{t_1+i_2} \pm \sum_1^{t_3} x^{\frac{1}{2}}_{t_1+t_2+i_3} \pm \cdots \pm \sum_1^{t_\nu} x^{\frac{1}{2}}_{t_1+t_2+\ldots+t_{\nu-1}+i_\nu} = 0 \tag{2}$$

where

$$t_1 + t_2 + \cdots + t_\nu = r + 1, \tag{3}$$

or

$$\sum_1^{\nu} \Phi_k = 0$$

where

$$\Phi_k = \sum_1^{t_k} x^{\frac{1}{2}}_{t_1+t_2+\ldots+t_{k-1}+i_k}.$$

3. Consider one of the $(t_k - 1)$-spaces, S_{t_k-1}, of the coördinate simplex, Δ_r, and let its equations be given by all the $r + 1 - t_k$ x's not contained in Φ_k equal to zero. This S_{t_k-1} is tangent to $V^{2^{r-1}}_{r-1}$ along a $V^{2^{t_k-1}}_{t_k-1}$ 2^{r-t_k}-uply. The equations of this $V^{2^{t_k-1}}_{t_k-1}$ of contact are the $r + 1 - t_k$ equations just mentioned and $\Phi_k = 0$. The nature of $V^{2^{t_k-1}}_{t_k-1}$ is identical with that of $V^{2^{r-1}}_{r-1}$ for $r = t_k - 1$. By allowing t_k to take on successively the values 2, 3, 4, $\cdots$, r, we find that the hypersurface is touched, respectively, by each of the $\binom{r+1}{2}$ edges of Δ_r in 2^{r-2} coincident points, by each of the $\binom{r+1}{3}$ planes of Δ_r along 2^{r-3} coincident conics, by each of the $\binom{r+1}{4}$ 3-spaces of Δ_r along 2^{r-4} coincident Steiner's quartic surfaces, $\cdots$, by each of the $r + 1$ hyperplane faces of Δ_r along a $V^{2^{r-2}}_{r-2}$.

4. Now the ν equations

$$\Phi_1 = 0, \quad \Phi_2 = 0, \cdots, \quad \Phi_\nu = 0 \tag{4}$$

represent ν hypersurfaces

$$V_{r-1}^{2t_1-1}, \quad V_{r-1}^{2t_2-1}, \cdots, \quad V_{r-1}^{2t_\nu-1}$$

respectively. The hypersurface $V_{r-1}^{2t_k-1}$ given by $\Phi_k = 0$ is the locus of the ∞^{t_k-2} S_{r+1-t_k}'s all passing through the S_{r-t_k} of Δ_r whose equations are obtained by equating to zero all the t_k x's contained in Φ_k and meeting the opposite S_{t_k-1} of Δ_r in the points of the variety $V_{t_k-2}^{2t_k-2}$ of contact between S_{t_k-1} and V_{r-1}^{2r-1}. The equations (4) taken simultaneously represent a $V_{r-\nu}^{2t_1+t_2+\cdots+t_\nu-2\nu}$ $\equiv V_{r-\nu}^{2r+1-2\nu}$ which is the variety of intersection of the ν hypersurfaces and is the locus of the $\infty^{r+1-2\nu}$ $S_{\nu-1}$'s incident with all the varieties of contact

$$V_{t_1-1}^{2t_1-2}, \quad V_{t_2-2}^{2t_2-2}, \cdots, \quad V_{t_\nu-2}^{2t_\nu-2}.$$

This variety $V_{r-\nu}^{2r+1-2\nu}$ is of multiplicity $2^{\nu-1}$ on V_{r-1}^{2r-1}. As the $r+1$ terms in the left-hand member of (1) can be grouped into ν groups as indicated in (2) in

$$N = (1/\nu!) \sum (r+1)!/(t_1! t_2! \cdots t_\nu!)$$

ways for all integral values of the t's greater than unity and satisfying relation (3), there are N $2^{\nu-1}$-fold varieties of dimension $r-\nu$ and order $2^{r+1-2\nu}$ on V_{r-1}^{2r-1}, constituting a composite $2^{\nu-1}$-fold variety of dimension $r-\nu$ and order

$$M = (2^{r+1-2\nu}/\nu!) \sum (r+1)!/(t_1! t_2! \cdots t_\nu!).$$

5. For a given of value of ν, the N $2^{\nu-1}$-fold $V_{r-\nu}^{2r+1-2\nu}$'s are of different types, depending on the values of the t's. Of course, for $\nu = 1$, we have V_{r-1}^{2r-1} itself. For $\nu = 2$, we find that the entire composite double variety on V_{r-1}^{2r-1} is of order $M = 2^{r-3}(2^r - r - 2)$ and is composed of $N = 2^r - r - 2$ ruled varieties all of order 2^{r-3}. These component varieties are distributed as follows: For t_1 or $t_2 = 2$, we have $\binom{r+1}{2} = \binom{r+1}{r-1}$ $(r-2)$-dimensional cones of order $r-3$ each having for vertex the point where an edge of Δ_r touches V_{r-1}^{2r-1} and standing on the V_{r-3}^{2r-3} of contact in the opposite S_{r-2} of Δ_r. Similarly, for t_1 or $t_2 = 3$, we have $\binom{r+1}{3} = \binom{r+1}{r-2}$ V_{r-3}^{2r-3}'s each of which has for generators the lines incident with the conic of contact in a plane of Δ_r and the V_{r-4}^{2r-4} of contact in the opposite S_{r-3} of Δ_r. Continuing in this manner by assigning all possible values to t_1, t_2 satisfying the relation $t_1 + t_2 = r+1$, we shall soon have obtained all the double varieties on the hypersurface V_{r-1}^{2r-1}.

6. We may proceed in a similar manner for the other values of ν, but we shall be content with an illustration. For $r = 5$, the double variety V_3^{100} of order 100 on V_4^{16} is composed of $N = 25$ V_3^{4}'s of which 15 are 3-dimensional cones each having its vertex on an edge of Δ_5 and standing on the Steiner's quartic surface of contact in the opposite S_3 of Δ_5 and 10 are ruled quartic varieties each being the locus of the ∞^2 lines incident with the two conics of contact lying in opposite planes of Δ_5. When $\nu = 3$, $t_1 = t_2 = t_3 = 2$, V_4^{16} has 15 quadruple planes each being determined by the points of contact on non-adjacent edges of the coördinate simplex.

7. On the composite double variety on $V_{r-1}^{2^{r-1}}$ is a pinch variety which is of dimension $r - 3$ and order $2^{r-4}r(r+1)$ and is composite, being composed of the $\binom{r+1}{2}$ $V_{r-3}^{2^{r-3}}$'s along which the S_{r-2}'s of Δ_r are tangent to $V_{r-1}^{2^{r-1}}$ doubly. Thus, the Steiner's quartic surface has six pinch points and the V_3^8 in S_4 has 10 pinch conics.

8. On the hypersurface $V_{r-1}^{2^{r-1}}$ there are numerous other multiple varieties whose multiplicities are not powers of 2. We shall here indicate a simple method by means of which these varieties may be obtained in any given case. Returning to the equation, (1), of the hypersurface, we find that the signs in its left-hand member may be combined in 2^r ways. Then, the left-hand member of the equation, after it is rationalized, is the product of the 2^r factors each with a different combination of signs. Now any point whose coördinates make q of these 2^r factors vanish is a q-fold point on the hypersurface. This method also enables us to find the $2^{\nu-1}$-fold varieties already mentioned above.

9. As an illustration, consider the V_3^8 in S_4. It has 10 double quadric cones $K^2{}_{ij}$ whose equations are

$$x_i : x_j : x_k : x_l : x_m = u^2 : u^2 : v^2 : (v-w)^2 : w^2$$

$[i, j, k, l, m = 1, 2, \cdots, 5]$. The right-hand members of these equations make the two factors

$$x_i^{\frac{1}{2}} - x_j^{\frac{1}{2}} + x_k^{\frac{1}{2}} - x_l^{\frac{1}{2}} - x_m^{\frac{1}{2}}, \qquad x_i^{\frac{1}{2}} - x_j^{\frac{1}{2}} - x_k^{\frac{1}{2}} + x_l^{\frac{1}{2}} + x_m^{\frac{1}{2}}$$

vanish. Now let $v = u$ and we have

$$x_i : x_j : x_k : x_l : x_m = u^2 : u^2 : u^2 : (u-w)^2 : w^2$$

which are the equations of 10 conics $C^2{}_{ijk}$, each being common to the three $K^2{}_{ij}$, $K^2{}_{jk}$, $K^2{}_{ki}$ and lying in the plane $x_i = x_j = x_k = 0$. As the coördinates

of the points on these conics make the three factors

$$-x_i^{1/2}+x_j^{1/2}+x_k^{1/2}-x_l^{1/2}-x_m^{1/2}, \qquad x_i^{1/2}-x_j^{1/2}+x_k^{1/2}-x_l^{1/2}-x_m^{1/2},$$
$$x_i^{1/2}+x_j^{1/2}-x_k^{1/2}-x_l^{1/2}-x_m^{1/2}$$

equal to zero, the conics are triple curves on V_3^8. Now putting $w=+2u$, we obtain five points whose coördinate $(1:1:1:1:4)$ make the four factors

$$+x_i^{1/2}-x_j^{1/2}-x_k^{1/2}-x_l^{1/2}+x_m^{1/2}, \qquad -x_i^{1/2}+x_j^{1/2}-x_k^{1/2}-x_l^{1/2}+x_m^{1/2},$$
$$-x_i^{1/2}-x_j^{1/2}+x_k^{1/2}-x_l^{1/2}+x_m^{1/2}, \qquad -x_i^{1/2}-x_j^{1/2}-x_k^{1/2}+x_l^{1/2}+x_m^{1/2}$$

vanish. These points are, therefore, quadruple points on the hypersurface V_3^8.

10. From what has been said concerning the singular varieties on the hypersurface $V_{r-1}^{2^{r-1}}$ it is not difficult to describe the singularities on any of its sections. In particular, the surface, $F^{2^{r-1}}$, in which a general S_3 meets it contains a composite double curve consisting of 2^r-r-2 components all of order 2^{r-3}. Of these component curves $r(r+1)/2$ are plane curves. Denote these by $C_{ij}^{2^{r-3}}$. They are the intersections of the given S_3 and the $r(r+1)/2$ $(r-2)$-dimensional double cones of order 2^{r-3} on $V_{r-1}^{2^{r-1}}$. They are all of deficiency $2^{r-5}(r-6)+1$, having $2^{r-5}(2^{r-2}-r)$ nodes. Each of these nodes is the common node of three curves $C_{ij}^{2^{r-3}}$, $C_{kl}^{2^{r-3}}$, $C_{mn}^{2^{r-3}}$ and is a quadruple point on $F^{2^{r-1}}$. There are in all

$$M_{v=3}=(2^{r-5}/3!)\sum(r+1)!/t_1!\,t_2!\,t_3!$$

such quadruple points where $t_1+t_2+t_3=r+1$. Now any three curves $C_{ij}^{2^{r-3}}$, $C_{ik}^{2^{r-3}}$, $C_{jk}^{2^{r-3}}$ which do not have a node in common meet in 2^{r-3} points on the line common to their planes. These points are triple points on $F^{2^{r-1}}$ and there are in all $r(r^2-1)2^{r-3}/6$ $[r>3]$ such triple points.

11. A general plane section, $c^{2^{r-1}}$, of $V_{r-1}^{2^{r-1}}$ has $2^{r-3}(2^r-r-2)$ nodes and $r+1$ r-fold tangents. Each of the latter is the intersection of the plane of $c^{2^{r-1}}$ and an S_{r-1} of Δ_r. As each r-fold tangent counts as $r(r-1)/2$ double tangents, $c^{2^{r-1}}$ has $2^{r-2}(2^{r-3}r^2-5r+8)-r(r^2-1)/2$ double tangents besides. The curve $c^{2^{r-1}}$ is of class $2^{r-2}r$ and is of deficiency $2^{r-3}(r-4)+1$.

12. All the previous results can be readily obtained from the following representation of the hypersurface upon a hyperplane R_{r-1}. Let $y_1, y_2, \cdots, y_{r+1}$ be the homogeneous point coördinates of R_{r-1} where

$$y_1+y_2+\cdots+y_{r+1}\equiv 0$$

and let the equation

$$u_1y_1^2 + u_2y_2^2 + \cdots + u_{r+1}y^2_{r+1} = 0 \tag{5}$$

define an ∞^r-system $|\,Q\,|$ of $(r-2)$-dimensional quadric varieties in R_{r-1}. Taking the members of $|\,Q\,|$ as fundamental varieties, we can represent the hypersurface $V^{2^{r-1}}_{r-1}$ upon R_{r-1}, the formulas of representation being

$$x_1 : x_2 : \cdots : x_{r+1} = y_1^2 : y_2^2 : \cdots y^2_{r+1}.$$

If we eliminate the y's from these equations, the result is equation (1). Each of the $(r-2)$-spaces y_i taken doubly is a member of $|\,Q\,|$.

13. By means of this representation we see that the hypersurface we are studying is of order 2^{r-1} as any $r-1$ general quadric varieties of $|\,Q\,|$ intersect in 2^{r-1} points. We now show that it is of class r. Consider the Jacobian varieties $J^{(1)}_{r-2}, J^{(2)}_{r-2}, \cdots, J^{(r-1)}_{r-2}$, all of order r, of any $r-1$ linear ∞^{r-1}-systems of $|\,Q\,|$. They all pass $h-1$ times through each of the $\binom{r+1}{h}$ S_{r-h-1}'s in which the $r+1$ S_{r-2}'s : y_i intersect h by h. Thus, each of the $\binom{r+1}{2}$ S_{r-3}'s in which the y's intersect two by two, each of the $\binom{r+1}{3}$ S_{r-4}'s in which the y's intersect three by three, $\cdots$, each of the $\binom{r+1}{r-2}$ lines in which the y's intersect $r-2$ by $r-2$, and finally, each of the $\binom{r+1}{r-1}$ points in which the y's intersect $r-1$ by $r-1$ are simple, double, $\cdots$, $(r-3)$-fold, and $(r-2)$-fold, respectively, on all the Jacobian varieties. Any two of these varieties, say $J^{(1)}_{r-2}, J^{(2)}_{r-2}$, intersect in a $V^{\binom{r}{2}}_{r-3}$ besides the $\binom{r+1}{2}$ S_{r-3}'s mentioned above. Now this $V^{\binom{r}{2}}_{r-3}$ intersects any third one, say $J^{(3)}_{r-2}$, in a $V^{\binom{r}{3}}_{r-4}$ besides the $\binom{r+1}{3}$ double S_{r-4}'s also mentioned above. Continuing in this manner, we find that the number of free points common to all the $r-1$ Jacobian varieties is $\binom{r}{r-1} = r$. Hence, the class of $V^{2^{r-1}}_{r-1}$ is r.

14. Of course, this fact can be easily derived from the equation of the hypersurface. Or, it may be seen from the fact that a general pencil of quadric varieties of $|\,Q\,|$ contains r conic varieties. Any general hyperplane S_{r-1} with the equation

$$u_1x_1 + u_2x_2 + \cdots + u_{r+1}x_{r+1} = 0$$

meets $V^{2^{r-1}}_{r-1}$ in a $V^{2^{r-1}}_{r-2}$ which corresponds to a quadric variety Q of $|\,Q\,|$. S_{r-1} will be a tangent hyperplane if Q is singular. The discriminant of equation (5) equated to zero, is, after dividing by $u_1u_2 \cdots u_{r+1}$,

$$1/u_1 + 1/u_2 + \cdots + 1/u_{r+1} = 0.$$

This is the equation of $V^{2^{r-1}}_{r-1}$ in hyperplane coördinates. Interpreting u_i as

point coördinates of S_r, we can regard this equation as representing a hypersurface V^r_{r-1} of order r and class 2^{r-1}, reciprocal to $V^{2^{r-1}}_{r-1}$. V^r_{r-1} has on it $r+1$ $(r-1)$-fold points, $\binom{r+1}{2}$ $(r-2)$-fold lines joining the $(r-1)$-fold points two by two, $\binom{r+1}{3}$ $(r-3)$-fold planes joining the $(r-1)$-fold points three by three, $\cdots$, $\binom{r+1}{r-1}$ simple S_{r-2}'s joining the $(r-1)$-fold points $r-1$ by $r-1$.

15. It is evident that V^r_{r-1} is representable upon R_{r-1} by means of the r-ic varieties of the type $J^{(1)}_{r-2}$, $J^{(2)}_{r-2}$, $\cdots$. Attention is here called to the involutorial r-ic transformation in S_r effected by means of r hyperquadric surfaces. To a point P we make correspond the point P' of intersection of its polar hyperplanes with respect to them. If P (or P') describes an S_{r-1} or in particular R_{r-1}, then P' (or P) describes an r-ic hypersurface. This hypersurface will be the V^r_{r-1} reciprocal to $V^{2^{r-1}}_{r-1}$ if the r hyperquadric surfaces have a self-polar simplex, Δ_r, in common. The intersection of Δ_r and R_{r-1} is the configuration, Δ'_r, formed by the $r+1$ $(r-2)$-spaces y_i. The quartic varieties in R_{r-1} coresponding to the ∞^r hyperplane sections of V^r_{r-1} pass through the $\binom{r+1}{r-1}$ $(r-3)$-spaces of Δ'_r and form an ∞^r-system. The Jacobian variety of a general linear ∞^{r-1}-family of this system is of order $r(r-1)$ but it is composed of the $r+1$ $(r-2)$-spaces y_i each taken $r-2$ times and a quadric variety whose equation is of the form (5). $r-1$ such Jacobian varieties yield $r-1$ quadric varieties having 2^{r-1} points in common, and, thus, we are led back to the representation of the hypersurface $V^{2^{r-1}}_{r-1}$ upon R_{r-1}.

16. If we take all the members of the $\infty^{(r+1)(r-1)/2}$-system of $(r-2)$-dimensional quadric varieties in R_{r-1} for fundamental varieties of representation, we can set up a one-to-one correspondence between the points of an $(r-1)$-dimensional variety of order 2^{r-1} in an $S_{(r+2)(r-1)/2}$ and the points of R_{r-1}. The projection upon an S_r of this variety from a general $S_{(r^2-r-4)/2}$ in $S_{(r+2)(r-1)/2}$ as center of projection is the $V^{2^{r-1}}_{r-1}$ we are investigating.

17. A general plane section $c^{2^{r-1}}$ of $V^{2^{r-1}}_{r-1}$ has for image the curve $C^{2^{r-1}}$ in which $r-2$ general quadric varieties of the system $|Q|$ in R_{r-1} intersect. If $C^{2^{r-2}}$ is projected on to an S_3, the projection will have $2^{r-3}(2^{r-2}-r+1)$ apparent double points. Hence $C^{2^{r-2}}$ is of deficiency $2^{r-3}(r-4)+1$ and, therefore, $c^{2^{r-1}}$ is of deficiency $2^{r-3}(r-4)+1$ as has already been mentioned.

THE UNIVERSITY OF CALIFORNIA.

TWO INVOLUTORIAL TRANSFORMATIONS, OF ORDERS 11 AND 9, ASSOCIATED WITH NULL RECIPROCITIES.

By VIRGIL SNYDER AND HAZEL E. SCHOONMAKER.

1. *Statement of the problem.* An interesting transformation may be defined as follows: Take four mutually skew lines a_i in space and a fixed point A not lying on any of them. Any plane through A cuts these lines in four points A_i. Then there exists in that plane a point O such that the lines OA, OA_i are projective with five given lines of a pencil. If the plane is allowed to describe the bundle A, the point O will describe a cubic surface * passing through A and containing the four lines a_i. This cubic surface is uniquely fixed. The analytic procedure will show that the converse is true. Hence we now have a $(1, 1)$ correspondence between the points of a cubic surface and the planes of a bundle through a point on it such that each point lies in its corresponding plane.

A $(1, 1)$ correspondence between two spaces such that the points of one space correspond to the planes of the other, each point lying in its corresponding plane, is called a null reciprocity. Montesano † has shown that such a transformation can be considered as the product of a birational transformation and a correlation, and that every birational transformation can be obtained as the product of a null reciprocity and a correlation.

The transformation described above is not good for all space. It will become so if, instead of keeping A fixed, we let A describe a line u. We then obtain a pencil of cubic surfaces $\psi + a\theta = 0$.

Any point P in space uniquely fixes a cubic of the pencil and a point A on u. The line AP meets the cubic associated with A in the point P', image of P. This transformation is involutorial.

2. *The parameters of the five lines OA_i.* Following analytically the steps outlined above, we first get the parameters of the five lines OA_i. Let the four skew lines be

$$a_1 \equiv x_3 = 0,\ x_1 + x_2 + x_4 = 0; \qquad a_2 \equiv x_4 = 0,\ x_1 - x_2 + x_3 = 0;$$
$$a_3 \equiv x_3 - x_4 = 0,\ x_1 + 2x_2 + x_3 = 0; \quad a_4 \equiv x_3 - 2x_4 = 0,\ 2x_1 + x_2 + x_4 = 0.$$

* Proved by Sturm, "Ueber correlative oder reciproke Bündel," *Mathematische Annalen*, Vol. 12 (1877), § 38.

† "Sulle reciprocità birazionali nulle dello spazio," *Rendiconti della reale Accademia dei Lincei*, Vol. 4 (1888), pp. 583-590.

Let $A \equiv (0, 0, a, 1)$ be any point on $u \equiv x_1 = 0,\ x_2 = 0$. Then any plane through A is

$$\pi \equiv c_1x_1 + c_2x_2 + c_3x_3 - ac_3x_4 = 0.$$

The plane $\pi = 0$ meets a_i in the points A, A_i. Let $O \equiv (y_1, y_2, y_3, y_4)$ and consider the coördinates of any point on OA_i. Cut these lines by the plane $x_1 = x_3$. This will give five points on the line $\pi = 0$, $x_1 = x_3$. Denote these points by B_i. For lack of space we omit the coördinates of these points.

The coördinates of any point on the line $\pi = 0$, $x_1 = x_3$ can be expressed linearly in terms of those of B, B_1. Then for a properly chosen h and k, $hB + kB_1 = B_i$. If we take B for $(1, 0)$, B_1 for $(0, 1)$, and put $h/k = \lambda_i$ for the parameter of B_i and hence of OA_i, then by solving the proper equations we find these parameters to be

$$\lambda_2 = \frac{c_1(y_1 + y_2 - y_3 + y_4) + c_2(y_1 + y_2 + y_3 + y_4) + 2c_3y_3}{ay_4 - y_1 + y_2 - y_3}$$

$$\lambda_3 = \frac{c_1(y_3 - 2y_1 - 2y_2 - 2y_4) + c_2(y_1 + y_2 + y_4) + c_3y_3(a - 1)}{ay_1 + 2ay_2 + ay_4 - y_1 - 2y_2 - y_3}$$

$$\lambda_4 = \frac{c_1(2y_1 + 2y_2 + 2y_4) - c_2(4y_1 + 4y_2 + 4y_4 - y_3) + c_3y_3(a - 1)}{4y_1 - 2ay_1 - ay_2 + y_2 + y_3 - ay_4}$$

3. *Equations of the pencil of cubic surfaces.* Let λ be the cross-ration of the first three and fourth lines; and λ' the cross-ratio of the first three and fifth lines. Then $\lambda = \lambda_3/\lambda_2$, $\lambda' = \lambda_4/\lambda_2$. Substituting the values of λ_i and remembering that O lies in $\pi = 0$, we see that O satisfies the following equations

$$\begin{aligned} c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 &= 0, \\ c_1\beta_1 + c_2\beta_2 + c_3\beta_3 &= 0, \\ c_1y_1 + c_2y_2 + c_3(y_3 - ay_4) &= 0, \end{aligned}$$

where α_i are functions of (y) and λ obtained by simplifying $\lambda = \lambda_3/\lambda_2$, and β_i are functions of (y) and λ'. Again we omit numerical details. For these equations to have a solution the determinant must vanish. Hence $\Delta = 0$ is the locus of 0. This determinant reduces to

$$\begin{vmatrix} \lambda(-2y_1-4y_2-2y_3)+2y_1+4y_2-y_3+3y_4 & y_3(-y_1+y_2-y_3)+y_4(3y_1+3y_2+3y_4) \\ \lambda'(8y_1+4y_2+2y_3)-8y_1-4y_2+y_3-6y_4 & y_3(-y_1+y_2-y_3)+y_4(-6y_1-6y_2-6y_4+3y_3) \end{vmatrix}$$

$$+ a \begin{vmatrix} \lambda(2y_1+4y_2+2y_4)+y_4-y_3 & y_3(-y_1+y_2-y_3)+y_4(3y_1+3y_2+3y_4) \\ \lambda'(-4y_1-2y_2-2y_4)+2y_4-y_3 & y_3(-y_1+y_2-y_3)+y_4(-6y_1-6y_2+3y_3-6y_4) \end{vmatrix}$$

$$= \psi + a\theta = 0.$$

4. *The complete intersection of two surfaces of the pencil.* The pencil $\psi + a\theta = 0$ contains A but not u, the locus of A, and it contains the four skew lines. Since the surfaces are cubics and contain four skew lines, they must contain their transversals, which are found to be

$$t_1 \equiv x_3 = 0,\ x_4 = 0;\quad t_2 \equiv 3x_1 + x_3 = 0,\ 2x_1 + x_2 + x_4 = 0.$$

These may be obtained by building the quadric surface containing a_1, a_2, a_3 and finding the points in which it is cut by a_4. The lines of the other regulus passing through these two points are the desired transversals. These lines do not meet u.

The residual intersection of any two cubics of the pencil is a space cubic curve c_3, $p = 0$ and of rank 4, meeting each a_i in two points, t_j in no points, and u in two points. If we find the points in which u meets $\psi = 0$ and $\theta = 0$ we find that u is a bisecant of c_3. The other properties of c_3 can be obtained by examining the map of a cubic surface.

5. *The equations of the transformation.* Any point on AP has coördinates of the form

$$\rho y'_1 = \sigma y_1,\ \rho y'_2 = \sigma y_2,\ \rho y'_3 = \sigma y_3 - \tau\psi,\ \rho y'_4 = \sigma y_4 + \theta\tau.$$

The line AP meets $F_3 = 0$ in the point P', image of P. We wish to determine σ and τ so that (y') lies on F_3. Substituting the above values in the expression for F_3, we obtain

$$F_3 \equiv \theta(y)\psi(y') - \psi(y)\theta(y') = 0.$$

Expanding by Taylor's series and recalling that A and (y) are on the surface, we find that

$$\sigma = \psi(y)\theta_2(y, \psi, \theta) - \theta(y)\psi_2(y, \psi, \theta),$$
$$\tau = \psi(y)\theta_1(y, \psi, \theta) - \theta(y)\psi_1(y, \psi, \theta),$$

where θ_i is the i-th polar of A as to θ, etc. More explicit values of σ and τ are rather complicated and we omit them here.

6. Order of the transformation and multiplicity of the fundamental curves. It is seen that σ is of order 10 and τ of order 8, hence the transformation is of order 11. Also σ is cubic in ψ and θ is quadratic in these quantities. It follows that the intersections of $\psi = 0$, $\theta = 0$, namely, $4a_i$, $2t_j$, c_3 are triple on the surfaces of the web. If A is any point on $u = 0$, it is found that the coördinates of A satisfy $\sigma = 0$, $\tau = 0$ and the first partial derivatives vanish for this point. Moreover,

$$\frac{\partial^2\sigma}{\partial y_i\partial y_k} = \psi(A)\,\frac{\partial^2\tau}{\partial y_i\partial y_k} \neq 0, \quad \text{and} \quad \frac{\partial^2 y'_i}{\partial y_i\partial y_k} = \frac{\partial^2\sigma}{\partial y_i\partial y_k} - \psi(A)\,\frac{\partial^2\tau}{\partial y_i\partial y_k} = 0.$$

Hence the surfaces $\sigma = 0$ and $\tau = 0$ have contact along a double line and this line is triple on the surfaces of the web.

In addition to these curves there are twenty parasitic lines. Eight of them, denoted by p_m, meet u, a_i, a_j, a_k; four, denoted by q, meet u, c_3 and the two transversals; and eight, denoted by g_{ij}, meet u, c_3, a_i, and t. To obtain p_4, for example, consider the F_2 containing a_1, a_2, and a_3, and the points in which it is met by u. Through each of these points is a line meeting a_1, a_2, a_3. Now consider the F_2 containing t_1, t_2 and u. This is met by c_3 in four points not on u. Through each of these is a line q. If we cut the pencil of cubics by a plane through a_i and t_j the residual is a line. For one cubic surface this line passes through the point in which u meets the plane as well as through the point in which c_3 meets the plane.

All these lines meet u. Let K be the point in which one of them g meets u. The cubic surface of the pencil belonging to K contains g, and by the transformation any point of g is transformed into the whole line g and into nothing else. Hence each of these lines is a simple fundamental line of the second kind.

7. *Images of the fundamental curves.* It is obvious that the image of $u = 0$ is $\sigma = 0$, and that any plane through u is invariant. Hence in any such plane π we have an involution. The plane π contains u and seven points $4A_i$, $2T_j$, $C_{3,1}$, the other two points $C_{3,2}$, and $C_{3,3}$ in which c_3 meets the plane being on u. The surface $\tau = 0$ is the surface of invariant points and its section by π is C_6: $4A_i{}^2 2T_j C^2{}_{3,1}$. Hence the involution in the plane is of the Geiser type. In this involution $A_i \sim C_3$: $A_i{}^2 3A_j 2T_k C_{3,1}$. If we revolve the plane about u, the Geiser involution will generate our space involution and hence the image of a_i is A_i: $a_i{}^2 3a_k 2t_j c_3 u$. Similarly $t_j \sim T_j$: $4a_i t_j{}^2 t_k c_3 u$. Since c_3 has two points on u its image is Γ_6: $4a_i 2t_j c_3{}^2 u^3$. Each of these surfaces contain some of the parasitic lines.

8. *Conclusion.* We find then that the transformation is given by

$$s_1 \sim s_{11}: \quad 4a_i{}^3 2t_i{}^3 c_3{}^3 u_3 8p_i 4q 8g_{ik}.$$

The jacobian is composed of

$$\begin{aligned} A_i &: \quad a_i{}^2 3a_k 2t_j c_3 u 8p_i 4g_{ij}, \\ T_j &: \quad 4a_i t_j{}^2 t_k c_3 u 4q 4g_{ij}, \\ \Gamma_6 &: \quad 4a_i 2t_j c_3{}^2 u^3 4q 8g_{ij}, \\ \sigma_{10} &: \quad 4a_i{}^3 2t_j{}^3 c_3{}^3 u_2 8p_i 4q 8g_{ij}, \end{aligned}$$

conjugates of a_i, t_j, c_3 and u. The surface of invariant points is

$$\tau: \quad 4a_i{}^2 2t_j{}^2 c_3{}^2 u_2 8p_i 4q 8g_{ij}.$$

This I_{11} contains two independent linear parameters λ, λ', hence there are really defined ∞^2 transformations. Any two distinct elements (λ, λ'), (λ'', λ''') generate an infinite discontinuous group of Cremona transformations, each if which contains the fundamental elements u, $4a_i$, $2t_j$; the residual space cubics form a congruence having u and $4a_i$ for bisecants which has been studied by Godeaux * and by J. de Vries.†

9. *The second transformation.* Another interesting transformation may be defined as follows: Take the lines u and $4a_i$ as before. A plane α through u meets each a_i in A_i. A point P in α defines a cross-ratio λ determined by the four ordered lines PA_i. The locus of P as λ remains constant and α turns about u is a quartic surface F_4: $u^2 4a_i$. In its equation, λ appears linearly, hence as λ varies, a pencil of quartic surfaces $F_4 \equiv \psi + \lambda\theta = 0$ is defined. By the method above we obtain

$$\begin{aligned}
\psi \equiv\ & x_2{}^2(x_2 - x_3 + 3x_4)(x_1 + 2x_2 - x_3 + 2x_4) \\
& + x_1x_2\,[(x_2 - x_3 + 3x_4)(x_1 + 2x_2 + x_4) \\
& + (-2x_1 + x_2 - 2x_3 + 3x_4)(x_1 + 2x_2 - x_3 + 2x_4)] \\
& + x_1{}^2(-2x_1 + x_2 - 2x_3 + 3x_4)(x_1 + 3x_2 + x_4), \\
\theta \equiv\ & x_2{}^2(4x_1 + 2x_2 + 2x_4)(2x_1 - 2x_2 + 2x_3 - 3x_4) \\
& + x_1x_2\,[(4x_1 + 2x_2 + 2x_4)(+x_1 - x_2 + x_3) \\
& + (4x_1 + 2x_2 - x_3 + 4x_4)(2x_1 - 2x_2 + 2x_3 - 3x_4) \\
& + x_1{}^2(4x_1 + 2x_2 - x_3 + 4x_4)(x_1 + 2x_2 + x_4).
\end{aligned}$$

Now take a second line u' not meeting any of the given ones nor lying on any surface of the pencil. A point (y) determines a surface of the pencil $F_4(y)$. The transversal of u, u' through (y) meets $F_4(y)$ in one residual point (y'). The transformation $y \sim y'$ is birational and involutorial.

Let u' be the line $x_1 - x_4 = 0$, $x_2 - 2x_3 = 0$. The plane determined by u and (y) meets u' in $U' \equiv (2y_1, 2y_2, y_2, 2y_1)$. Any point on the line $U'(y)$ has coördinates of the form

$$\rho x_1 = (\sigma + 2\tau)y_1, \ \rho x_2 = (\sigma + 2\tau)y_2, \ \rho x_3 = \sigma y_3 + \tau y_2, \ \rho x_4 = \sigma y_4 + 2\tau y_1,$$

and

$$F_4(y) \equiv \psi(x)\theta(y) - \psi(y)\theta(x) = 0.$$

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* "Sur une congruence linéo-linéaire de cubiques gauches, *Bulletins de la Classe des Sciences, Académie Royale de Belgique, 1809*, p. 531.

† The congruence of twisted cubics that cut five lines twice," *Proceedings Koninklijke Akademie van Wetenscappen te Amsterdam*, Vol. 31 (1928), pp. 454-458.

For the point (y') on this line we have

$$\tau = -2\,[\theta(y)\psi(\bar{y}, y) - \psi(y)\theta(\bar{y}, y)],$$
$$\sigma = \theta(y)\psi(\bar{y}) - \theta(\bar{y})\psi(y),$$

wherein $(\bar{y}) \equiv U'$, and $\psi(\bar{y}, y)$ is the polar of U' as to the quadric coefficients of x_1^2, $2x_1x_2$, x_2^2 in ψ, etc.

10. *The fundamental elements.* The transformation is of order 9. In $\sigma = 0$, u appears to order 6, and in $\tau = 0$, to order 5. Since τ is multiplied by y_1 or y_2 in each equation of the transformation the conjugates of the planes of space contain u^6.

From the definition of the transformation, every plane through u remains invariant. In each plane of this pencil the transformation is perspective Jonquières, of order 3, having U' for double point and $4A_i$ for simple fundamental points. Every conic of the pencil $A_1 \cdots A_4$ is transformed into itself except that one passing through U' which has the point U' for its conjugate. Hence u' is double on I_9. The locus of this conic as α turns about u is $\sigma = 0$.

The line a_i is simple on I_9, since the conjugate of any point A_i on it is the line $U'A_i$. The locus of this line is the quadric $uu'a_i \equiv H_i$.

The lines meeting $ua_1a_2a_3 = p_4$, etc. are basis lines of the pencil of quartics, since u is a double line. The plane up_i meets u' in T'. Three points A_k lie on p_i and A_i does not. Consider any point (y) in this plane. The line joining it to A_i meets p_i in a point which determines λ, and the line $T'(y)$ meets p_i in its conjugate (y'). Thus the conjugate of p_i is the plane up_i. For one value of λ, the line $(y)A_i$ passes through T'. These lines are parasitic, or simple fundamental lines of the second kind.

Finally, consider the transversals of u, u', a_i, $a_k \equiv q_{ik}$. The plane u, q_{ik} meets u' in Q' on q_{ik}, hence each line q_{ik} is a simple fundamental line of the second kind. This completes the configuration of fundamental elements. The conjugate of u is $\sigma + 2\tau = 0$, and $\tau = 0$ is the locus of invariant points. The surfaces $\sigma = 0$, and $\tau = 0$ touch each other along the line u. Every plane through u' is invariant; in each plane is a perspective Jonquières I_7: $U^6 4A_i 8Q_k$.

The table of characteristics has the form

$$s_1 \sim s_9 : u^6u'^2 4a_i 8p_i 12q_{ik} 8t_i,$$
$$u \sim U_8 : u^5u'^2 4a_i 8p_i 12q_{ik} 8t_i,$$
$$u' \sim U'_8 : u^6u' 4a_i 8p_i 12q_{ik} 8t_i,$$
$$a_i \sim H_{2i} : uu'a_i 6q_{ik} 2t_i,$$
$$p_i \sim \pi_i : up_it_i,$$
$$J_{32} \equiv U_8U'_8 4H_2,\ 8\pi_i.$$

SOME PROPERTIES OF THE FUNDAMENTAL CURVES OF A BIRATIONAL TRANSFORMATION IN SPACE.

By MARIAN M. TORREY.

Given a web of homaloidal surfaces $|\phi|$ in space (x), homogeneous of degree n in x_1, x_2, x_3, x_4, and forming a regular system, that is, one having no fixed tangent planes at any isolated fundamental point, or at any point of a fundamental curve. A $(1,1)$ correspondence is defined between spaces (x) and (y) by the equations

$$y_1 : y_2 : y_3 : y_4 = \phi_1 : \phi_2 : \phi_3 : \phi_4,$$
$$x_1 : x_2 : x_3 : x_4 = \psi_1 : \psi_2 : \psi_3 : \psi_4,$$

where the ψ's are homogeneous of degree m in y_1, y_2, y_3, y_4, and also form a regular system. To a line of (x) corresponds a curve C'_n in (y), and to a line of (y) corresponds a curve C_m in (x). Suppose the web $|\phi|$ has a fundamental curve u of the first kind, order ν, multiplicity k. Then it is known * that if all surfaces of the web tangent to a fixed plane π at a point P of u are considered, they form a net which corresponds to a bundle of planes in (y), with vertex at a point P'. As π turns about the line through P tangent to u, the point P' generates a curve v_P of order k, which is the image of P, and as P moves along u, v_P generates a surface of degree equal to the number of variable points in which any C_m meets u. Every plane through P corresponds to a ψ passing simply through the curve v_P.† The planar neighborhood π of P corresponds to the spatial neighborhood of P'.

The object of the present discussion is to study this correspondence of directions at P and P' in more detail. It will be shown that a direction l at P in π corresponds to a planar neighborhood at P' in π'_l, a plane through the line tangent to v_P at P': as l rotates about P in π, π'_l revolves about the tangent to v_P. A direction at P' in π'_l determines a point in the second neighborhood of P on a curve C_m tangent to l at P: stated a little differently,

* D. Montesano, "Su la teoria generale delle corrispondenze birazionali fra i punti dello spazio," *Atti della R. Accademia di Napoli*, Ser. 2, Vol. 17 (1927), No. 8, p. 4.

† L. Godeaux, "Sur les courbes fondamentales des transformations birationelles de l'espace," *Bulletins de l' Académie royale de Belgique*, Ser. 5, Vol. 15 (1929), pp. 317-318.

any C_m tangent to l corresponds to a line of π'_l; the osculating plane of C_m at P corresponds to a surface tangent to π'_l at P', and having the line corresponding to C_m for an inflexional tangent.

It is clear first that the surface ψ corresponding to π, which may be called ψ_π, has a double point at P', for since every curve C_m which is the intersection of two surfaces of the net has one more intersection with π than with any other plane through P, all lines through P' have one more intersection with ψ_π than with other ψ's containing v_P. Hence any pencil of planes with axis l through P in π corresponds to a pencil of surfaces through v_P, all having a fixed tangent plane at P', namely π'_l, which passes through the line tangent to v_P. As l rotates about P, remaining in π, the plane π'_l changes position, always containing the tangent to v_P at P'.

There are ∞^1 pencils of surfaces ϕ tangent to π at P, each having its curve of intersection C_m tangent to l at P. The condition for this is that every plane through l should cut two surfaces of the pencil defining C_m in two plane curves, which in addition to having k-fold points at P, and the contact along one branch which determines the net, have contact of the second order along this branch, that is, in terms of cartesian coördinates, have equal second derivatives at P. This is a linear condition on the parameters of the net, reducing the net to a pencil. Furthermore the condition varies with the particular value imposed on the second derivative, giving ∞^1 such pencils.

Every plane through l cuts such a C_m in two points at P, and one plane through l, the osculating plane of C_m, meets it in three points at P. In space (y), therefore, every surface tangent to π'_l touches the line corresponding to such a C_m, so that this line must be in the plane π'_l. One surface tangent to π'_l at P', the one corresponding to the osculating plane of C_m, has this line for its inflexional tangent.

For an accurate picture of this relationship between the lines of π'_l and the surfaces ψ tangent to π'_l, note first that each line of π through P corresponds to a composite curve in (y) made up of v_P and a variable curve C'_{n-k}, these two curves intersecting at P'. A generic plane through P' meets this composite curve in two points at P'; any plane of the pencil with axis tangent to v_P meets the curve in three points at P'; the plane tangent to both v_P and C'_{n-k} meets the curve in four points at P'. Hence any line of π through P meets a general surface of the net tangent to π in $k+1$ points at P, and a surface corresponding to a plane through the tangent to v_P in $k+2$ points. That is, such a surface has two sheets tangent to π, or if v_P is a straight line so that u is simple, a surface corresponding to a plane through v_P has

a double point at P, since in this case such a surface must belong to every net defined by a fixed tangent plane at P. Furthermore the surface which corresponds to the plane π'_l tangent to both v_P and C'_{n-k}, is met by the corresponding l in $k+3$ points at P, that is, such an l must be an inflexional tangent for one of the sheets of this surface tangent to π, or in the case when u is simple, l must be tangent to the surface.

Now consider the curves C_m tangent to a particular l of π and corresponding to the lines of π'_l through P'. All such curves lie on the surface corresponding to π'_l, which may be called ϕ_l. Any curve osculating a particular plane α of the pencil through l must have three consecutive points in the plane α at P, and these points must also be on ϕ_l. Since ϕ_l has two sheets tangent to π, there are two curves on which the points can approach P, remaining in α and on ϕ_l, and one of these curves osculates l, since one sheet of ϕ_l osculates l. Hence two C_m osculate each α, one osculating l, and the other osculating the second branch tangent to l of the section of ϕ_l by α.

If another C_m osculated this plane, it would have two points consecutive to P in common with one of the above curves (C_m). But two C_m can intersect in only one variable point.

The C_m osculating l osculates every plane through l. Hence each plane α through l is osculated by two and only two curves C_m through P, one peculiar to α, and one osculating every plane through l. It follows that one line of π'_l is an inflexional tangent for every surface of the pencil tangent to π'_l; this line is also tangent to the proper image of l, a C'_{n-k}. The second inflexional tangent in π'_l varies with the surface of the pencil.

When k equals one, a similar argument shows that there is only one curve C_m osculating each plane through l, since only one branch of the section of ϕ_l by a plane through l is tangent to l. This curve varies with the plane through l. The pencil of surfaces in (y) tangent to π'_l has one variable inflexional tangent at P', the other being the line v_P.

Hence there is a $(1,1)$ correspondence between directions at P in π, and planar neighborhoods at P' defined by the planes passing through the tangent to v_P, and there is a $(1,1)$ correspondence between neighborhoods of the second order at P defined by the osculating planes of curves C_m tangent to the same line of π, and directions at P' in the corresponding planar neighborhood.

Now suppose the $|\phi|$ have a fundamental curve of the second kind, u, order ν, multiplicity $k\nu'$, which corresponds to a fundamental curve of the second kind v on the web $|\psi|$, order ν', multiplicity $k\nu$, where k is a positive integer. Then if the condition is imposed that some surfaces of the web

in (x) be tangent at P, a point of u, to an arbitrary plane π_1 through the tangent line to u, a net of surfaces results all tangent at P to π_1 and to $k-1$ other planes $\pi_2, \cdots, \pi_k$, and these k planes form an involution, any one determining the other $k-1$.* Furthermore the net so determined has k fixed tangent planes at every point of u. This net corresponds to a bundle of planes with vertex at a point of v, say P'. As the set of k planes through P varies, the point P' generates v. A similar situation exists for nets $|\psi|$ tangent to a fixed plane at a point of v. All ψ tangent to one fixed plane at a point of v are tangent to $k-1$ others, and as the set of k planes varies, the point remaining fixed, the vertex of the corresponding bundle of planes in (x) generates the curve u.

At a generic point P of u, determine a net $|\phi|$ tangent at P to π_1. This fixes a set of planes $\pi_1, \pi_2, \cdots, \pi_k$, whose equations are

$$\sum_{j=1}^{4} a_{ij}x_j = 0; \qquad (i = 1, 2, \cdots, k).$$

Call the vertex of the corresponding bundle of planes in (y) P', and consider the particular set of k planes through the tangent to v at P' which determines the point P of u. It will be shown that the planes $\pi_1, \cdots, \pi_k$, can be paired uniquely with the k planes at P' in such a way that directions in π_i at P correspond to directions in π'_i at P', and vice versa. In other words, whatever the value of k, any tangent plane to u uniquely determines a tangent plane to v, such that the two planar neighborhoods correspond in the transformation. It follows that as P varies on u, the net $|\phi|$ being fixed, P' remains fixed, and the k planes π'_i, corresponding to the planes tangent to $|\phi|$ at P, generate the pencil whose axis is the tangent to v at P'. The spatial neighborhood of P' is thus shown to correspond to ∞^1 planar neighborhoods defined by the tangent planes to the fixed net $|\phi|$ at points of u.

The plane π_1 is tangent to all surfaces of the net $|\phi|$ corresponding to the bundle of planes, vertex P'. Hence any line l in π_1 through P meets any surface of the net in $kv'+1$ points at P. The curve C'_n corresponding to such a line is composite. One component is a curve $C'_{n-kv'}$, the proper image of l, and the other is the curve v counted k times.* Since l meets any surface of the net in $n-kv'-1$ variable points, $C'_{n-kv'}$ meets every plane of the bundle at P'. It must now be shown that all $C'_{n-kv'}$ corre-

* D. Montesano, "Sulla teoria generale delle corrispondenze birazionali dello spazio," *R. Accademia dei Lincei Atti. Rendiconti*, Ser. 5, Vol. 30 (1921), pp. 447-451.

sponding to lines of π_1, and therefore lying on the surface $\sum_{j=1}^{4} a_{1j}\psi_j = 0$, are tangent to one and only one of the k planes through P'.

The line l meets a generic plane of (x) in one variable point, and a generic plane of the bundle, vertex P, at P. The curve $C'_{n-kv'}$ meets any ψ of the web in one variable point, and in kv' fixed points at P'. It must therefore meet any surface of the net corresponding to the bundle of planes, vertex P, in $kv'+1$ points at P'. This can happen only if it is tangent at P' to one of the planes tangent to the net $|\psi|$. It cannot be tangent to more than one, for if tangent to i of the k planes, $C'_{n-kv'}$ would have $kv'+i$ points of intersection at P with every surface of the net, which would mean i points of intersection at P for l and every plane of the bundle. Furthermore, as l varies in π_1, turning about P, $C'_{n-kv'}$ varies continuously, always passing through P', so that it is not possible to have some of these $C'_{n-kv'}$ tangent to one of the k planes π'_i at P', and others tangent to another. Call the plane to which these $C'_{n-kv'}$ are tangent π'_1, and let its equation be

$$\sum_{j=1}^{4} b_{1j}y_j = 0.$$

Since it has been shown that $C'_{n-kv'}$ has two intersections at P' with π'_1, l must have $kv'+2$ intersections at P with the surface corresponding to π'_1, namely $\sum_{j=1}^{4} b_{1j}\phi_j = 0$, and since l is any line of π_1, the section of this surface by π_1 has a $kv'+2$-fold point at P.

It is shown in a similar way that the section of the surface $\sum_{j=1}^{4} a_{1j}\psi_j = 0$ by π'_1 has a $kv+2$-fold point at P', while the section of any other surface of the net by π'_1 has a $kv+1$-fold point at P'. For a line l' through P' in π'_1 corresponds to a curve C_{m-kv} which passes through P and is tangent to one of the k common tangent planes of the net $|\phi|$. That this plane is π_1 is evident from the fact that l' meets a C'_n on $\sum_{j=1}^{4} a_{1j}\psi_j = 0$, which is composed of v taken k times and the proper image $C_{n-kv'}$, in two consecutive points at P'. Hence the curve C_m, the complete image of l', meets l, the image of C'_n, in two consecutive points at P, so that l is a line in the plane tangent to u and C_{m-kv}, the components of C_m. But l is any line through P in $\pi_1 \equiv \sum_{j=1}^{4} a_{1j}x_j = 0$, and hence C_{m-kv} is tangent to π_1. The remainder of the argument is identical.

Furthermore the correspondence must be unique. For suppose lines of

a second plane π_2 in (x) corresponded to $C'_{n-kv'}$ which also were tangent to the plane π'_1. Then lines in the plane π'_1 would correspond to curves C_{m-kv} tangent to both π_1 and π_2, and this has been shown to be impossible. Hence π_2 is paired with another of the k planes in (y) which we call π'_2. Proceeding thus, the k planes in (y) can be named $\pi'_1, \pi'_2, \cdots, \pi'_k$.

We now consider whether there is a correspondence between directions through P in π_i, and directions through P' in π'_i. Take a pencil of planes in the bundle, vertex P, with axis in π_i,

$$\sum_{j=1}^{4} c_j x_j + \lambda \sum_{j=1}^{4} a_{ij} x_j = 0,$$

where $\sum c_j x_j = 0$ is satisfied by the coördinates of P. In (y) space the corresponding pencil of surfaces has the equation

$$\sum_{j=1}^{4} c_j \psi_j + \lambda \sum_{j=1}^{4} a_{ij} \psi_j = 0,$$

all members of the pencil being tangent to the planes $\pi'_1, \cdots, \pi'_k$. Take a section of this pencil by the plane π'_i. A pencil of plane curves results with a $kv+1$-fold point at P'. All tangents to this pencil at P' are fixed, since the section of $\sum_{j=1}^{4} a_{ij}\psi_j = 0$ by π'_i has been shown to have a $kv+2$-fold point at P', and therefore this equation has vanishing derivatives of order $kv+1$. This was to have been expected, since the curve of intersection of surfaces of the pencil has, in addition to the fundamental curve v, kv-fold for the system, two other branches through P' tangent to π'_i, namely a curve v_i consecutive to v, and the curve $C'_{n-kv'}$, the proper image of a line of π_i through P. This means that surfaces of a pencil in (y) corresponding to planes of a pencil with axis in π_i, have contact of the first order at P' along the sheets tangent to $\pi'_1, \pi'_2, \cdots, \pi'_{i-1}, \pi_{i+1}, \cdots, \pi'_k$, and contact of the second order at P' along the sheets tangent to π'_i.* The necessary though not sufficient condition for such contact is that the $kv+1$ inflexional tangents to the surfaces at P' in π'_i shall be fixed for the pencil. $kv-1$ tangents coincide with the tangent line to v, leaving two fixed directions in π'_i distinct from this line. A section of this pencil of surfaces by any other plane π'_j will not give fixed tangents at P' in that plane, and any pencil of planes with axis not in π_i, corresponds to a pencil of surfaces which does not have fixed tangents at P' in π'_i.

* E. Pascal, *Repertorium der Höheren Mathematik*, Vol. II_2, p. 654.

Similarly if the bundle of planes in (y) is considered, and a pencil of the bundle determined with axis in π'_i, a pencil of curves is found in π_i with two fixed tangents at P distinct from the tangent to u. However it cannot be said that the direction in π'_i given by the axis of the pencil of planes corresponds to the directions given by the two fixed inflexional tangents in π_i of the corresponding pencil of surfaces. The correspondence in directions at P and P' in the planes π_i and π'_i must be considered in two ways. One approaches P in π_i either along a curve C_{m-kv}, image of a line l' of π'_i, or along the tangent l to that curve. These two directions, apparently one, give different approaches in π'_i to P'. If P is approached along C_{m-kv}, P' is approached along a line l' of π'_i which is one of the inflexional tangents of the pencil of surfaces whose curve of intersection, $C'_{n-kv'}$, corresponds to l. If P is approached along l, P' is approached along $C'_{n-kv'}$, which is not in general tangent to l'.

If neither u nor v is a straight line, there are two curves C_{m-kv} tangent to each l of π_i, and two curves $C'_{n-kv'}$ tangent to each l' of π'_i. However, if one of the fundamental curves, say v, is a straight line, there is only one curve C_{m-kv} tangent to each l, but two curves $C'_{n-kv'}$ tangent to each l'; if both u and v are straight lines, there is one C_{m-kv} tangent to each l, and one $C'_{n-kv'}$ tangent to each l'.

A simple example of a fundamental curve of the second kind for which $k=2$ is given by Miss Hudson.* u and v are both straight lines, and the transformation, a cubic involution, is defined by the equations

$$x_1 : x_2 : x_3 : x_4 = y_1y_4^2 : y_2y_3^2 : y_3y_4^2 : y_3^2y_4.$$

v is the line $y_3 = y_4 = 0$. The system is not perfectly regular, since it has fixed tangent planes at the points $(1,0,0,0)$ and $(0,1,0,0)$, but these points may be avoided. As all known transformations which have fundamental curves of the second kind with $k > 1$ are products of transformations for which $k = 1$, so in this case the cubic is the product of the two quadratics

$$x_1 : x_2 : x_3 : x_4 = z_1z_3 : z_2z_4 : z_3^2 : z_3z_4,$$
$$z_1 : z_2 : z_3 : z_4 = y_1y_4 : y_2y_3 : y_3y_4 : y_3^2.$$

Consider a point on v, $(\xi, \eta, 0, 0)$. If the condition is imposed that at this point the plane $my_3 = y_4$ shall be tangent to the web

$$ay_1y_4^2 + by_2y_3^2 + cy_3y_4^2 + dy_3^2y_4 = 0,$$

* H. Hudson, *Cremona Transformations in Plane and Space*, Cambridge University Press (1927).

a second tangent plane is fixed, namely $my_3 = -y_4$, and the condition $a = -\eta b/\xi m^2$ is imposed on a and b. Hence the web becomes the net

$$b(\eta y_1 y_4^2 - \xi m^2 y_2 y_3^2) + c y_3 y_4^2 + d y_3^2 y_4 = 0,$$

which corresponds to the bundle of planes with vertex at $(\xi m^2, \eta, 0, 0)$.

Similarly if we take the web of surfaces in (x), the tangent planes at $(\xi m^2, \eta, 0, 0)$ are given by the equation

$$b\eta x_3^2 + a\xi m^2 x_4^2 = 0.$$

If one tangent plane is taken as $x_3 = mx_4$, the other is determined as $x_3 = -mx_4$, and $a = -\eta b/\xi$, so that the web is again reduced to a net

$$b(\eta x_1 x_4^2 - \xi x_2 x_3^2) + c x_3 x_4^2 + d x_3^2 x_4 = 0,$$

which corresponds to the bundle of planes with vertex at $(\xi, \eta, 0, 0)$. Any line through $(\xi, \eta, 0, 0)$ in the plane $my_3 = y_4$ of (y) has for image, in addition to the line $x_3 = x_4 = 0$ counted twice, a line through $(\xi m^2, \eta, 0, 0)$ in the plane $x_3 = mx_4$. This property of the transformation was noted by Miss Hudson in another connection.* In this simple case the curve C_{m-kv} referred to above is a straight line; hence the problem of distinguishing between the direction at P in π_i as defined by C_{m-kv} and by the tangent to C_{m-kv} does not arise.

A simple non-involutorial transformation where u and v are not both straight lines can be obtained by applying a quadratic transformation to the above cubic. Thus if we combine

$$x_1 : x_2 : x_3 : x_4 = z_1 z_4^2 : z_2 z_3^2 : z_3 z_4^2 : z_3^2 z_4$$

with

$$z_1 : z_2 : z_3 : z_4 = y_1 g : y_2 g : y_3 g : f,$$

where g is linear and f quadratic in $y_1, \cdots, y_4$, and in addition $f = 0$ is satisfied by the coördinates $(0, 0, 0, 1)$, there results a quintic

$$x_1 : x_2 : x_3 : x_4 = y_1 f^2 : y_2 y_3^2 g^2 : y_3 f^2 : y_3^2 gf$$

with the double conic $y_3 = f = 0$ fundamental of the second kind. The plane $g = 0$ cannot in this case be chosen as $y_4 = 0$, since that gives contact conditions all along the fundamental curve in (x) space. In any case, the web has fixed tangent planes at the points $y_2 = y_3 = f = 0$ and $y_1 = y_3 = f = 0$ which are distinct from $(0, 0, 0, 1)$, and at the points $y_3 = g = f = 0$, and these points must be avoided in applying the theory.

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* H. Hudson, *loc. cit.*, p. 271.

It is easily seen that the inverse of the quintic is a sextic transformation which has the four-fold line $x_3 = x_4 = 0$, image of the fundamental curve $y_3 = f = 0$.

If a tangent plane π_i is fixed at a point P of the conic $y_3 = f = 0$, there is determined a tangent plane π'_i at a point P' of the line $x_3 = x_4 = 0$. A line l in π_i through P corresponds to a curve C'_3 tangent to π'_i at P'. A curve of the system tangent to l at P corresponds to a line in π'_i, which is an inflexional tangent common to the pencil of surfaces defining the C'_3 above, but this inflexional tangent of the surfaces is not tangent to C'_3: in other words, the line l and the curve tangent to l define a common direction at P, but their images define two distinct directions at P'.

The essential difference between the two cases arises from the fact that tangency to a fixed plane at a point P of a fundamental curve of the second kind entails tangency at every point of the curve, while for fundamental curves of the first kind this is not true. Hence there is contact of the second order for certain pencils of surfaces if P is on a fundamental curve of the second kind, and contact of the first order if the curve is of the first kind; hence also one, or at most two, curves of the system tangent to each line through P if u is of the second kind, and ∞^1 curves of the system tangent to each line through P if u is of the first kind.

ON SURFACES POSSESSING A NET OF PLANE ISOTHERMALLY. CONJUGATE CURVES.*

By C. A. NELSON.

1. *Introduction.* Let $y^{(k)} = y^{(k)}(u, v)$, $z^{(k)} = z^{(k)}(u, v)$, $(k = 1, 2, 3, 4)$ be the homogeneous coördinates of the two focal surfaces S_y, S_z—which we assume to be distinct—of any congruence, Γ, of lines. Let u and v be the parameters of those curves on S_y and S_z along which the developables of the congruence touch the focal surfaces. Then the congruence may be characterized, projectively, by a completely integrable system of partial differential equations of the form

$$\text{(D)} \qquad \begin{aligned} y_v &= mz, \qquad z_u = ny, \\ y_{uu} &= ay + bz + cy_u + dz_v, \\ z_{vv} &= a'y + b'z + c'y_u + d'z_v, \end{aligned}$$

in which the coefficients are analytic functions of u and v and the subscripts denote differention.†

The integrability conditions of system (D) are

$$\text{(I)} \qquad \begin{aligned} &c = f_u,\ d' = f_v,\ b = -d_v - df_v,\ a' = -c'_u - c'f_u,\ mn - c'd = f_{uv}, \\ &m_{uu} + d_{vv} + df_{vv} + d_v f_v - f_u m_u = ma + db', \\ &n_{vv} + c'_{uu} + c'f_{uu} + c'_u f_u - f_v n_v = c'a + nb', \\ &2m_u n + mn_u = a_v + f_u mn + a'd, \\ &m_v n + 2mn_v = b'_u + f_v mn + bc', \end{aligned}$$

where f may be any analytic function of u and v.

The form of the equations (D) is unchanged by the group of transformations

$$\text{(T)} \qquad y = \lambda(u)\bar{y}, \quad z = \mu(v)\bar{z}, \quad \bar{u} = \phi(u), \quad \bar{v} = \psi(v),$$

λ, μ, ϕ and ψ being arbitrary functions. Although the form of (D) is unaltered, the coefficients themselves are changed. Among the combinations of these coefficients which are relatively invariant we cite

* Presented to the Society, December 27, 1928.

† E. J. Wilczynski, "Sur la théorie générale des congruences," *Mémoires publiés par la Classe des Sciences de l'Académie Royale de Belgique*, Deuxième serie, tome 3 (1911). This memoir will be referred to as the Brussels Paper.

$$(1.1)\qquad c',\ d,\ m,\ d_1 = d\Big(c'd - \frac{\partial^2}{\partial u \partial v}\log d\Big),\quad m_1 = m\Big(mn - \frac{\partial^2}{\partial u \partial v}\log m\Big).$$

Three of the covariants are

$$(1.2)\qquad \rho = y_u - (m_u/m)y,\quad \sigma = z_v - (n_v/n)z,\quad \epsilon = \rho_u - (\partial/\partial u)\log m_1 \cdot \rho.$$

The covariant σ, when z is replaced by $z^{(k)}$, defines the minus first Laplace transform of S_z. The surfaces S_σ and S_z are the focal surfaces of the minus first transform of Γ. Similarly, S_ρ and S_ϵ are the focal surfaces of the plus first transform of Γ.

If we wish to concentrate upon the focal surface S_y rather than upon the congruence Γ, we obtain a characterizing system of equations by eliminating z from (D). The result is

$$(Y)\qquad \begin{aligned} y_{uu} &= (d/m)y_{vv} + f_u y_u - (d/m)(f_v + m_v/m + d_v/d)y_v + ay,\\ y_{uv} &= (m_u/m)y_{vv} + mny. \end{aligned}$$

The parametric curves upon S_y form a conjugate system. When it happens that this system has equal invariants and consists entirely of plane curves, the surface S_y is greatly restricted. It is the purpose of this paper to investigate the implications of these restrictions.

2. *Analytic formulation.* It is well known that if the $u =$ const. curves are plane the envelope of the tangents to these curves is developable or degenerate. The invariantive condition for this is *

$$(2.1)\qquad c' = 0.$$

Similarly, the $v =$ const. curves are plane if, and only if,

$$(2.2)\qquad d_1 = 0.$$

Finally, the conjugate system has equal invariants when, and only when,

$$(2.3)\qquad (\partial^2/\partial u \partial v)\log m = 0.\dagger$$

It should be remarked that the above conditions are also necessary and sufficient for a plane isothermally-conjugate net on S_y.‡

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* Brussels Paper, p. 28. G. M. Green, "Projective Differential Geometry of One-parameter Families of Space Curves, and Conjugate Nets on a Curved Surface," *American Journal of Mathematics*, Vol. 37 (1915), p. 236.

† E. J. Wilczynski, "The General Theory of Congruences," *Transactions of the American Mathematical Society*, Vol. 16 (1915), p. 319. This paper will be cited as Congruence Paper.

‡ Congruence Paper, p. 322.

A comparison of the conditions (2.1), (2.2), (2.3) with (1.1) indicates that they may be replaced by

$$(2.4) \qquad c' = 0, \quad (\partial^2/\partial u \partial v) \log d = 0, \quad (\partial^2/\partial u \partial v) \log m = 0.$$

We shall assume throughout that $d \neq 0$, $m \neq 0$. Geometrically this means that the surface S_y is neither developable nor degenerate.*

Since no transformation (T) disturbs the form of (D) we seek a transformation that will simplify the conditions (2.4). Now †

$$(2.5) \qquad \bar{m} = (\mu/\lambda)(1/\psi_v) m, \quad \bar{d} = (\mu/\lambda)(\psi/\phi_u^2) d,$$

so that if we write, from (2.4), $d = U(u) \cdot V(v)$ and $m = U_1(u) \cdot V_1(v)$ the transformation for which

$$\lambda = U_1, \quad \mu = 1/(VV_1)^{\frac{1}{2}}, \quad \phi_u = (U/U_1)^{\frac{1}{2}}, \quad \psi_v = (V_1/V)^{\frac{1}{2}}$$

reduces both $\bar{m}$ and $\bar{d}$ to unity. Having made this transformation the conditions (2.4) may be replaced by

$$(2.6) \qquad c' = 0, \quad d = m = 1.$$

The most general transformation not violating (2.6) is given by

$$(2.7) \qquad \lambda = k_1, \quad \mu = k_1 k_2, \quad \phi_u = \psi_v = k_3,$$

where k_1, k_2 are constants.

The above values for c', d, m greatly simplify the integrability conditions. They become

$$(2.8) \quad c = f_u, \quad d' = f_v, \quad b = -f_v, \quad a' = 0, \quad n = f_{uv}, \quad b' = f_{vv} - a,$$
$$(2.9) \quad n_{vv} - f_v n_v = nb', \quad n_u = a_v + f_u n, \quad 2n_v = b'_u + f_v n.$$

The expressions (2.6) and (2.8) give all of the coefficients of (D) in terms of a, f, and their derivatives. The expressions (2.9) represent conditions upon these two coefficients. Eliminating b' and n from (2.9) we obtain

$$(2.10) \qquad a_v = f_{uuv} - f_u f_{uv}, \quad a_u = f_v f_{uv} - f_{uvv}, \quad a_{uv} = f_{uv} a.$$

We may sum up the results in the theorem: *Let a non-developable, non-degenerate surface S_y be referred to a conjugate system of plane curves with equal invariants. Then the congruence of tangents to the $u =$ constant curves may be characterized, projectively, by*

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* Brussels Paper, p. 28.

† Brussels Paper, pp. 20, 23.

$$
\begin{aligned}
&y_v = z, \quad z_u = f_{uv} y, \\
(\Delta) \qquad &y_{uu} = ay - f_v z + f_u y_u + z_v, \\
&z_{vv} = (f_{vv} - a) z + f_v z_v,
\end{aligned}
$$

where a, f satisfy the conditions (2. 10).

3. *An alternate form for* (2. 10). We introduce the quantities

$$h = f_{uv}, \quad h_1 = f_{uv} - (\partial^2/\partial u \partial v) \log f_{uv}. \tag{3. 1}$$

Solving the first two equations of (2. 10) for f_{uuv}, f_{uvv} and differentiating, we find

$$f_{uuvv} = a_{vv} + f_{uv}^2 + f_u (f_v f_{uv} - a_u), \tag{3. 2}$$

$$f_{uvvu} = -a_{uu} + f_{uv}^2 + f_v (a_v + f_u f_{uv}). \tag{3. 3}$$

whence $h^2 h_1$ has the two expressions

$$
\begin{aligned}
h^2 h_1 &= -a_u a_v + f_{uv} (f_v a_v - a_{vv}), \\
&= -a_u a_v + f_{uv} (a_{uu} - f_u a_u).
\end{aligned}
$$

Using the first form

$$(h^2 h_1)_u = f_u h^2 h_1.$$

Similarly, the second gives

$$(h^2 h_1)_v = f_v h^2 h_1,$$

whence

$$h^2 h_1 = k e^f,$$

with k an arbitrary constant. Since $k \neq 0$—unless $h h_1 = 0$, a special case we return to later—a transformation (T) can be found that will reduce k to unity and not violate the conditions (2. 7).

From the first and last of (2. 10) we find

$$a = f_{uuuv}/f_{uv} - f_u f_{uuv}/f_{uv} - f_{uu}. \tag{3. 4}$$

Eliminating a from the first two

$$[f_{uu} + f_{vv} - (f_u^2 + f_v^2)/2]_{uv} = 0. \tag{3. 5}$$

The conditions (3. 4), (3. 5) and

$$h^2 h_1 = e^f \tag{3. 6}$$

are equivalent to (2. 10).

4. *Integration of* (Y). We turn now to the system (Y) which becomes

$$
\begin{aligned}
&y_{uu} = y_{vv} + f_u y_u - f_v y_v + ay, \\
&y_{uv} = f_{uv} y
\end{aligned} \tag{4. 1}
$$

and proceed to obtain the general integral of $y_{uv} = f_{uv}y$. This equation has equal invariants

$$h = k = f_{uv}. \tag{4.2}$$

Its first Laplace transform has the invariants

$$h_1 = f_{uv} - (\partial^2/\partial u \partial v) \log f_{uv}, \quad k_1 = h, \tag{4.3}$$

while, for the second transform,

$$h_2 = (f - \log h^2 h_1)_{uv}, \quad k_2 = h_1. \tag{4.4}$$

A glance at (3.6) shows that $h_2 = 0$, so that according to the general theory,* the general solution of $y_{uv} = f_{uv}y$ is

$$y = -aU + f_u U' + U'' + aV + f_v V' + V'', \tag{4.5}$$

in which U and V are arbitrary functions of u alone and v alone, respectively. The primes denote differentiation.

This value of y must also satisfy the first equation in (4.1). Direct substitution gives

$$\begin{aligned} &U^{IV} + (2f_{uu} - f_u{}^2 - 2a)U'' \\ &\quad + (f_{uuu} - f_u f_{uu} - a_u)U' + (-a_{uu} + a_{vv} + a_u f_u - a_v f_v + a^2)U \\ &\equiv V^{IV} + (2f_{vv} - f_v{}^2 + 2a)V'' \\ &\quad + (f_{vvv} - f_v f_{vv} + a_v)V' + (a_{vv} - a_{uu} - a_v f_v + a_u f_u + a^2)V. \end{aligned} \tag{4.6}$$

The coefficients of U'' and V'' are functions are functions of u alone and v alone, respectively, as (2.10) shows. Define

$$2U_0 \equiv 2f_{uu} - f_u{}^2 - 2a, \quad 2V_0 \equiv 2f_{vv} - f_v{}^2 + 2a. \tag{4.7}$$

Then equate the two expressions (3.2), (3.3). We find

$$a_{uu} + a_{vv} - f_u a_u - f_v a_v = 0 \tag{4.8}$$

so that the coefficient of U—and also of V—may be written indifferently

$$-2a_{uu} + 2a_u f_u + a^2, \quad 2a_{vv} - 2a_v f_v + a^2.$$

Differentiate the first form with respect to v, whence (2.10) shows the result to be zero. Similarly, the derivative of the second form with respect to u is zero. Thus the coefficient of U is a constant k and the condition (4.6) becomes

$$U^{IV} + 2U_0 U'' + U'_0 U' + kU \equiv V^{IV} + 2V_0 V'' + V'_0 V' + kV. \tag{4.9}$$

* Darboux, *Théorie Générale des Surfaces*, t. 2, Chap. 2.

Differentiating we find

$$U^{V} + 2U_0U''' + 3U'_0U'' + (U''_0 + k)U' = 0, \tag{4.10}$$

$$V^{V} + 2V_0V''' + 3V'_0V'' + (V''_0 + k)V' = 0, \tag{4.11}$$

both of which are self-adjoint ordinary differential equations.

We summarize our results: *The homogeneous point coördinates of the most general, non-degenerate, non-developable surface S_y referred to a conjugate net of plane curves with equal invariants are given by*

$$y^{(i)} = -aU_i + f_uU'_i + U''_i + aV_i + f_vV'_i + V''_i, \qquad (i = 1, 2, 3, 4),$$

where a and f are solutions of (2.10) *and U_i, V_i satisfy* (4.9).

5. *Integration of* (4.10). The equation (4.10) is of odd order and is also self-adjoint. Darboux has shown * that such an equation can always be expressed in the form

$$f[(d/du)g(U)] = 0, \tag{5.1}$$

where, in the present instance,

$$\begin{aligned} f &\equiv \lambda U + \lambda_1U' + \lambda_2U'', \\ g &\equiv \mu U + \mu_1U' + \mu_2U'', \end{aligned} \tag{5.2}$$

and are adjoint expressions, that is,

$$\mu = \lambda - \lambda'_1 + \lambda_2'', \quad \mu_1 = -\lambda_1 + 2\lambda'_2, \quad \mu_2 = \lambda_2. \tag{5.3}$$

We desire to identify the equations (5.1) and (4.10). Comparison of coefficients of U^{V} indicates that we may take $\lambda_2 = \mu_2 = 1$. Whence from (5.3), $\mu_1 = -\lambda_1$ and $\mu = \lambda - \lambda'_1$. Direct calculation gives

$$\begin{aligned} f[(d/du)g(U)] \equiv\ & U^{V} + (2\lambda - 4\lambda'_1 - \lambda_1^2)U''' \\ & + 3(\lambda' - 2\lambda_1'' - \lambda_1\lambda'_1)U'' \\ & + (3\lambda'' - 4\lambda_1''' - 3\lambda_1\lambda_1'' + 2\lambda_1\lambda' + \lambda^2 - 2\lambda\lambda'_1)U' \\ & + [\lambda''' - \lambda_1^{IV} + \lambda_1(\lambda'' - \lambda_1''') + \lambda(\lambda' - \lambda_1'')]\,U. \end{aligned} \tag{5.4}$$

From the coefficients of U''', U'', U',

$$U_0 = \lambda - 2\lambda'_1 - \lambda_1^2/2, \quad k = (\lambda - \lambda'_1)^2 + 2(\lambda - \lambda'_1)'' + 2\lambda_1(\lambda - \lambda'_1)' \tag{5.5}$$

Replace U_0 by

$$U_0 = k^{1/2} - (d^2/du^2)\log U_1 - \tfrac{1}{2}(U'_1/U_1)^2 \tag{5.6}$$

* *Théorie des Surfaces*, t. 2, p. 134.

where U_1 is a new function of u. Finally, choose

$$\lambda_1 = U'_1/U_1, \quad \lambda = k^{\frac{1}{2}} + d^2 \log U_1/du^2, \tag{5.7}$$

whence

$$\begin{aligned} f(U) &\equiv (k^{\frac{1}{2}} + d^2 \log U_1/du^2)U + (U'_1/U_1)U' + U'', \\ g(U) &\equiv k^{\frac{1}{2}}U - (U'_1/U_1)U' + U''. \end{aligned} \tag{5.2'}$$

Darboux gives another form that is more useful in the sequel. We write

$$\begin{aligned} f(U) &\equiv (1/\alpha)(d/du)(1/\beta)(d/du)\alpha\beta \cdot U, \\ g(U) &\equiv \alpha\beta(d/du)(1/\beta)(d/du)(U/\alpha). \end{aligned} \tag{5.8}$$

Differentiating out and identifying with (5.2') we find

$$\beta'/\beta = U'_1/U_1 - 2\alpha'/\alpha, \quad \alpha'' - (U'_1/U_1)\alpha' + k^{\frac{1}{2}}\alpha = 0.$$

Therefore the self-adjoint equation (4.10) may be written in the form (5.1) with f and g given by (5.8) and

$$\beta = U_1/\alpha^2, \tag{5.9}$$

$$\alpha'' - (U'_1/U_1)\alpha' + k^{\frac{1}{2}}\alpha = 0. \tag{5.10}$$

We may write down immediately five independent integrals of (4.10). They are

$$\begin{aligned} &u_1 = \alpha, \quad u_2 = \alpha \int \beta du, \quad u_3 = \alpha \int \beta du \int \frac{du}{\alpha\beta}, \\ &u_4 = \alpha \int \beta du \int \frac{du}{\alpha\beta} \int \frac{du}{\alpha\beta}, \quad u_5 = \alpha \int \beta du \int \frac{du}{\alpha\beta} \int \frac{du}{\alpha\beta} \int \beta du, \end{aligned} \tag{5.11}$$

where each integral includes those that follow. The integrals are to be regarded as having a common fixed lower limit u_0 and a common variable upper limit u.

In a similar fashion we write five independent integrals of (4.11).

$$\begin{aligned} &v_1 = \gamma, \quad v_2 = \gamma \int \delta\, dv, \quad v_3 = \gamma \int \delta\, dv \int \frac{dv}{\gamma\delta}, \\ &v_4 = \gamma \int \delta\, dv \int \frac{dv}{\gamma\delta} \int \frac{dv}{\gamma\delta}, \quad v_5 = \gamma \int \delta\, dv \int \frac{dv}{\gamma\delta} \int \frac{dv}{\gamma\delta} \int \delta\, dv, \end{aligned} \tag{5.12}$$

where

$$\delta = V_1/\gamma^2, \quad V_0 \equiv k^{\frac{1}{2}} - d^2 \log V_1/dv^2 - \tfrac{1}{2}(V'_1/V_1)^2, \tag{5.13}$$

$$\gamma'' - (V'_1/V_1)\gamma' + k^{\frac{1}{2}}\gamma = 0, \tag{5.14}$$

with corresponding conventions with respect to the integrals.

A pair u_i, v_j, when substituted in (4. 5), will give a solution of (Y) provided

$$F(u_i) \equiv G(v_j), \tag{5.15}$$

in which $F(U)$ is a notation for the left member of (4. 9) and $G(V)$ for the right member. We verify that

$$\begin{aligned} &F(u_1)=0,\ F(u_2)=0,\ F(u_3)=k^{\frac{1}{2}},\ F(u_4)=\alpha'/U_1+k^{\frac{1}{2}}\int_{u_0}^{u}\frac{\alpha}{U_1}\,du,\\ &F(u_5)=\frac{U_1}{\alpha'}\int_{u_0}^{u}\frac{U_1}{\alpha^2}\,du+k^{\frac{1}{2}}\int_{u_0}^{u}\frac{\alpha}{U_1}\,du\int_{u_0}^{u}\frac{U_1}{\alpha^2}\,du+1/\alpha, \end{aligned} \tag{5.16}$$

with similar results for $G(v_j)$. We need merely to replace α, U_1, u by γ, V_1, v. $F(u_4)$ and $F(u_5)$ are constants for their derivatives are zero. Hence

$$F(u_4)=\alpha'(u_0)/U_1(u_0),\quad F(u_5)=1/\alpha(u_0).$$

In order that

$$F(u_i)=G(v_i) \qquad (i=1,2,3,4,5), \tag{5.17}$$

we agree that $\alpha'(u_0)=U_1(u_0)$, $\alpha(u_0)=\gamma(v_0)=1$, $\gamma'(v_0)=V_1(v_0)$. Hence (5. 10) may be written in the equivalent form

$$\alpha'/U_1+k^{\frac{1}{2}}\int_{u_0}^{u}(\alpha/U_1)\,du=1.$$

These five pairs (u_i, v_i) give five solutions $y^{(i)}$ which, of course, cannot be independent. We select four pairs so as to obtain four linearly independent $y^{(i)}$ from (4. 5). The following may be used

$$\begin{aligned} &\bar{u}_1=2u_1,\quad \bar{u}_2=2u_3, && \bar{u}_3=u_4,\quad \bar{u}_4=2u_5,\\ &\bar{v}_1=v_1,\quad \bar{v}_2=v_3+k^{\frac{1}{2}}v_4,\quad && \bar{v}_3=v_4,\quad \bar{v}_4=v_4+v_5. \end{aligned} \tag{5.18}$$

In fact, $F(\bar{u}_i)=G(\bar{v}_i)$, $(i=1,2,3,4)$ and the determinant

$$|\,y_{vv}^{(i)}, y_u^{(i)}, y_v^{(i)}, y^{(i)}\,|$$

is not identically zero. This latter fact may be verified most conveniently by using the particular solution

$$a=-4/(u^2-v^2),\quad f=\log 2/(u^2-v^2)^2. \tag{5.19}$$

with which we may use

$$u_0=0,\quad U_1=1, \tag{5.20}$$

whence $\alpha=u+1$. In this case the above determinant has the value $8/(u^2-v^2)^2$.

The closing phrase in the theorem at the end of paragraph 4 should now be amended to read "and $U_i = \bar{u}_i$, $V_i = \bar{v}_i$" where $\bar{u}_i$ and $\bar{v}_i$ are given by the expressions (5.18), (5.11), and (5.12).

6. *Exceptional case* $h_1 = 0$, $h \neq 0$. The Liouville equation $h_1 = 0$ has for its general solution

$$f = \log 2U'_0V'_0/U_1V_1(U_0 + V_0)^2, \tag{6.1}$$

where the U's and V's are arbitrary functions of u alone and v alone, respectively. From the integrability conditions (2.10) the value of a_{uv} may be computed from either of the first two conditions. Then the third gives

$$a = \partial^2 \log U_1/\partial u^2 + (U'_1/U_1)(\partial/\partial u) \log 2U'_0V'_0/(U_0 + V_0)^2,$$

or

$$a = -\partial^2 \log V_1/\partial v^2 - (V'_1/V_1)(\partial/\partial v) \log 2U'_0V'_0/(U_0 + V_0)^2,$$

whence, equating,

$$[U'_1(2U'_0V'_0)/U_1(U_0 + V_0)^2]_u = -[V'_1(2U'_0V'_0)/V_1(U_0 + V_0)^2]_v. \tag{6.2}$$

Hence there exists a function $\phi(u, v)$ such that

$$\phi_v = U'_1(2U'_0V'_0)/U_1(U_0 + V_0)^2, \quad \phi_u = -V'_1(2U'_0V'_0)/V_1(U_0 + V_0)^2.$$

Integrating the first and substituting the result in the second, we obtain

$$\begin{aligned} d^2 \log U_1/du^2 &+ (U'_1/U_1)[U_0''/U'_0 - U'_0/(U_0 + V_0)] \\ &- U'_2(U_0 + V_0)/2U'_0 - V'_1V'_0/V_1(U_0 + V_0) = 0. \end{aligned} \tag{6.3}$$

In the same way the integration of the second gives

$$\begin{aligned} d^2 \log V_1/dv^2 &+ (V'_1/V_1)[V_0''/V'_0 - V'_0/(U_0 + V_0)] \\ &+ V'_2(U_0 + V_0)/2V'_0 - U'_1U'_0/U_1(U_0 + V_0) = 0. \end{aligned} \tag{6.4}$$

In these equations U_2, V_2 are arbitrary functions entering through the integrations. From (6.2), (6.3), (6.4)

$$U'_2/U'_0 = V'_2/V'_0 = \text{constant} = k,$$

whence

$$U_2 = kU_0 + k_3, \quad V_2 = kV_0 + 2k_1 + k_3,$$

and

$$\phi = -U'_1(2U'_0)/U_1(U_0 + V_0) + kV_0 + k_3,$$

or

$$\phi = V'_1(2V'_0)/V_1(U_0 + V_0) + kV_0 + 2k_1 + k_3.$$

By equating these two expressions for ϕ we find

$$(6.5)\qquad \begin{aligned} &2U'_0U'_1/U_1 - kU_0^2 + 2k_1U_0 \\ &\quad = -2V'_0V'_1/V_1 - kV_0^2 - 2k_1V_0 = \text{constant} = -k_2. \end{aligned}$$

Differentiating with respect to u and dividing by $2U'_0$

$$d^2 \log U_1/du^2 + U'_1U_0''/U_1U'_0 - kU_0 + k_1 = 0.$$

In a similar manner

$$d^2 \log V_1/dv^2 + V'_1V_0''/V_1V'_0 + kV_0 + k_1 = 0.$$

The value of a may now be written, using the last three equations,

$$(6.6)\qquad (U_0 + V_0)a = kU_0V_0 + k_1(U_0 - V_0) + k_2.$$

We have thus obtained the formulas for the functions a and f. They are given by (6.6) and (6.1). The values of U_1, V_1 which enter therein are found from (6.5). The arbitraries thus consist of U_0, V_0 and five constants.

Let us now investigate the problem of integrating the system (Y). Since the Laplace invariant h_1 of the second equation is zero, the general integral is

$$(6.7)\qquad y = (h_u/h)U + U' + (h_v/h)V + V',$$

in which U, V are arbitrary functions of u alone and v alone. Direct substitution in the first equation gives

$$\begin{aligned} (6.8)\qquad &U''' + (U'_1/U_1)U'' + [2\{U_0, u\} - d^2 \log U_1/du^2]\, U' \\ &\quad + [\{U_0, u\}' + 2(U'_1/U_1)\{U_0, u\} + d^3 \log U_1/du^3]\, U \\ &= V''' + (V'_1/V_1)V'' + [2\{V_0, v\} - d^2 \log V_1/dv^2]V' \\ &\quad + [\{V_0, v\}' + 2(V'_1/V_1)\{V_0, v\} + d^3 \log V_1/dv^3]\, V, \end{aligned}$$

where we use the familiar notation $\{U_0, u\}$ for the Schwarzian derivative of U_0 with respect to u. Differentiating with respect to u

$$\begin{aligned} (6.9)\qquad &U^{IV} + (U'_1/U_1)U''' + 2\{U_0, u\}U'' + [3\{U_0, u\}' + 2(U'_1/U_1)\{U_0, u\}]\, U' \\ &\quad + [\{U_0, u\}'' + (U'_1/U_1)\{U_0, u\}'] = 0, \end{aligned}$$

since

$$(U'_1/U_1)\{U_0, u\}' + 2(d^2 \log U_1/du^2)\{U_0, u\} + d^4 \log U_1/du^4 = 0.$$

There is a similar equation for the function V.

Although the equation (6.9) is not self-adjoint it may be written in the form

$$(6.9')\qquad (1/U_1)(d/du)U'_0U_1(d/du)(1/U'_0)(d/du)(1/U'_0)(d/du)U'_0U = 0.$$

Hence four linearly independent solutions are

$$(6.10)\quad u_1 = 1/U'_0,\quad u_2 = U_0/U'_0,\quad u_3 = U_0^2/2U'_0,$$
$$u_4 = u_1\int_{u_0}^{u}(u_3/U_1)\,du - u_2\int_{u_0}^{u}(u_2/U_1)\,du + u_3\int_{u_0}^{u}(u_1/U_1)\,du.$$

Similarly, we may use

$$(6.11)\quad v_1 = 1/V'_0,\quad v_2 = -V_0/V'_0,\quad v_3 = V_0^2/2V'_0,$$
$$v_4 = v_1\int_{v_0}^{v}(v_3/V_1)\,dv - v_2\int_{v_0}^{v}(v_2/V_1)\,dv + v_3\int_{v_0}^{v}(v_1/V_1)\,dv,$$

as a set of linearly independent solutions of the V-equation corresponding to (6.9).

Call the left member of (6.8) $F(U)$ and the right member $G(V)$. Then

$$(6.12)\quad \begin{array}{llll} F(u_1)=k, & F(u_2)=k_1, & F(u_3)=-k_2/2, & F(u_4)=1/U_1(u_0),\\ G(v_1)=-k, & G(v_2)=-k_1, & G(v_3)=k_2/2, & G(v_4)=1/V_1(v_0), \end{array}$$

the k's entering through (6.5).

In order to obtain four linearly independent $y^{(i)}$ that will be solutions of the equations (4.1) we select pairs of linear expressions $\bar{u}_i$, $\bar{v}_i$ of u_i and v_i, respectively, such that $F(\bar{u}_i) = G(\bar{v}_i)$ while the determinant

$$|\ y_{vv}^{(i)},\ y_{uu}^{(i)},\ y_v^{(i)},\ y^{(i)}\ | \neq 0,$$

the $y^{(i)}$ being computed from (6.7). To accomplish this we take

$$(6.13)\quad \begin{array}{ll} \bar{u}_1 = u_1 + a_1u_2, & \bar{v}_1 = v_1 + \alpha_1v_2,\\ \bar{u}_2 = u_2 + a_2u_3, & \bar{v}_2 = v_2 + \alpha_2v_3,\\ \bar{u}_3 = u_3 + a_3u_4, & \bar{v}_3 = v_3 + \alpha_3v_4,\\ \bar{u}_4 = u_4 + a_4u_3, & \bar{v}_4 = v_4 + \alpha_4v_3, \end{array}$$

where

$$(6.14)\quad a_1+\alpha_1 = -2k/k_1,\ a_2+\alpha_2 = 4k_1/k_2,\ a_3 = \alpha_3 + k_2 \neq 0,\ a_4+\alpha_4 = 0.$$

Hence

$$y^{(1)} = -4/(U_0+V_0) + 2k(U_0-V_0)/k_1(U_0+V_0),$$
$$y^{(2)} = -2(U_0-V_0)/(U_0+V_0) + 4k_1U_0V_0/k_2(U_0+V_0),$$
$$(6.15)\quad y^{(3)} = 2U_0V_0/(U_0+V_0) + k_2\Big(\frac{2}{U_0+V_0}\int_{v_0}^{v}\frac{v_3}{V_1}\,dv - \frac{U_0-V_0}{U_0+V_0}\int_{v_0}^{v}\frac{v_2}{V_1}\,dv - \frac{U_0V_0}{U_0+V_0}\int_{v_0}^{v}\frac{v_1}{V_1}\,dv\Big)$$
$$y^{(4)} = -\frac{2}{U_0+V_0}\Big(\int_{u_0}^{u}\frac{u_3}{U_1}\,du + \int_{v_0}^{v}\frac{v_3}{V_1}\,dv\Big) + \frac{U_0-V_0}{U_0+V_0}\Big(\int_{u_0}^{u}\frac{u_2}{U_1}\,du + \int_{v_0}^{v}\frac{v_2}{V_1}\,dv\Big) + \frac{U_0V_0}{U_0+V_0}\Big(\int_{u_0}^{u}\frac{u_1}{U_1}\,du + \int_{v_0}^{v}\frac{v_1}{V_1}\,dv\Big).$$

The above determinant is not identically zero for it does not vanish when $a = 0$.

We turn now to some geometrical considerations. From (1.2)

$$\epsilon = y_{uu} - (h_u/h)y_u, \quad \sigma = z_v - (h_v/h)z, \tag{6.16}$$

so that

$$\sigma = \epsilon - ay - (f_u - h_u/h)y_u + (f_v - h_v/h)z. \tag{6.17}$$

By direct computation

$$\begin{aligned} \sigma_u &= 0, \quad \sigma_v = (f_v - h_v/h)\sigma, \\ \epsilon_v &= 0, \quad \epsilon_u = (f_u - h_u/h)\epsilon, \end{aligned} \tag{6.18}$$

from which we see that the developables S_z and S_ρ reduce to cones whose vertices are the fixed points P_σ and P_ϵ, respectively. Ordinarily these vertices do not coincide. The relation (6.17) shows this to be the case only if

$$a = 0, \quad f_u - h_u/h = 0, \quad f_v - h_v/h = 0$$

which conditions are equivalent to the single one

$$a = 0. \tag{6.19}$$

The surfaces for which $a = 0$ have been studied elsewhere.* We note that the formulas (6.15) are still valid. We merely allow k, k_1, k_2 to approach zero in such a way that k/k_1 and k_1/k_2 approach zero. In this case the cones reduce to coincident quadric cones. In fact,

$$\begin{aligned} z^{(1)} &= 4V'_0/(U_0 + V_0)^2, \quad z^{(2)} = 4U_0V'_0/(U_0 + V_0)^2, \\ z^{(3)} &= 2U_0{}^2V'_0/(U_0 + V_0)^2, \end{aligned}$$

while

$$\begin{aligned} \rho^{(1)} &= 4U'_0/(U_0 + V_0)^2, \quad \rho^{(2)} = -4U'_0V_0/(U_0 + V_0)^2, \\ \rho^{(3)} &= 2U'_0V_0{}^2/(U_0 + V_0)^2, \end{aligned}$$

whence both $z^{(i)}$ and $\rho^{(i)}$ satisfy

$$2x_1x_3 - x_2{}^2 = 0, \tag{6.20}$$

which is a quadric cone with vertex at $(0, 0, 0, 1)$.

It should be noted that these coördinates refer to an arbitrary tetrahedron of reference. To find the equation of the cone referred to the local tetra-

* Lane, "Conjugate Systems with Indeterminate Axis Curves," *American Journal of Mathematics*, Vol. 43 (1921), p. 52. The totality of surfaces possessing conjugate systems with indeterminate axis curves forms, to be sure, a larger class than those we are considering here.

hedron whose vertices are P_y, P_z, P_ρ, P_σ we express the derivatives of z in terms of y, z, ρ, σ. Those which we need are

$$z_u = f_{uv}y, \qquad z_{uu} = f_{uuv}y + f_{uv}\rho,$$
$$z_{uv} = f_{uvv}y + f_{uv}z, \quad z_{vv} = (f_{vv} + f_v^2)z + f_v\sigma.$$

Placing these values in the expansion

$$\begin{aligned} Z &= z + z_u \cdot \Delta u + z_v \cdot \Delta v \\ &\quad + \tfrac{1}{2}(z_{uu} \cdot \Delta u^2 + 2z_{uv} \cdot \Delta u \Delta v + z_{vv} \cdot \Delta v^2) + \cdots \\ &= z_1 y + z_2 z + z_3 \rho + z_3 \sigma \end{aligned} \tag{6.21}$$

we have

$$\begin{aligned} z_1 &= f_{uv} \cdot \Delta u + (f_{uuv}/2)\Delta u^2 + f_{uvv} \cdot \Delta u \Delta v + \cdots, \\ z_2 &= 1 + \cdots, \\ z_3 &= (f_{uv}/2)\Delta u^2 + \cdots. \end{aligned} \tag{6.22}$$

In each of the above expressions the omitted terms are of higher order than those given.

Since the vertex of the cone is at $(0, 0, 0, 1)$ and the tangent planes at $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$ are $x_3 = 0$ and $x_2 = 0$, respectively, the equation must be of the form

$$y_1^2 - 2\lambda y_2 y_3 = 0. \tag{6.23}$$

To determine λ we insert the values of z in (6.23) and demand that all terms of the second order in Δu and Δv disappear. Hence

$$\lambda = f_{uv} = 2U'_0V'_0/(U_0 + V_0)^2.$$

7. *Exceptional Case* $h = f_{uv} = 0$. When $f_{uv} = 0$, the integrability conditions (I) give

$$f = U_0 + V_0, \qquad a = \text{constant} = \alpha. \tag{7.1}$$

The general solution of the second equation of (Y) is

$$y = U + V, \tag{7.2}$$

whence substitution in the first gives

$$U'' - U'_0U' - \alpha U \equiv V'' - V'_0V' + \alpha V = \text{constant}. \tag{7.3}$$

For a moment we turn to geometrical considerations. The fundamental equations (Δ) show that

$$z_u = 0, \quad z_{vv} = (f_{vv} - \alpha)z + f_v z_v,$$

so that the developable S_z becomes a straight line. Similarly, for S_ρ. These two lines do not meet in general. In fact, any point on the line L_z is given by

$$z_v + \kappa z = y_{vv} + \kappa y_v, \tag{7.4}$$

while an arbitrary point of L_ρ is of the form

$$\rho_u + \lambda\rho = y_{uu} + \lambda y_u = y_{vv} + (\lambda + U'_0)y_u - V'_0 y_v + \alpha y. \tag{7.5}$$

For intersection all second order determinants of the matrix

$$\begin{pmatrix} 1 & 0 & \kappa & 0 \\ 1 & \lambda + U'_0 & -V'_0 & \alpha \end{pmatrix}$$

must vanish. Consequently,

$$\alpha = 0, \quad \lambda = -U'_0, \quad \kappa = -V'_0,$$

and we may say that *a necessary and sufficient condition for the lines* L_z, L_ρ to meet is that $a = \alpha = 0$.

Now for the integration of the equation (7.3). Differentiation with respect to u yields

$$U''' - U'_0 U'' - (U_0'' + \alpha)U' = 0. \tag{7.6}$$

Replace the arbitrary function U_0 by

$$U'_0 = U_1''/U'_1 - \alpha U_1/U'_1, \tag{7.7}$$

so that $U = U_1$ is a particular integral of (7.6). Then make the usual transformation $U' = U'_1\bar{U}$ and integrate. In this way we find three independent integrals of (7.6) to be

$$u_1 = 1, \quad u_2 = U_1, \quad u_3 = U_1 \int_{u_0}^{u} (U_2/U'_1)\,du - \int_{u_0}^{u} (U_1 U_2/U'_1)\,du, \tag{7.8}$$

where

$$U_2 = \exp\left[-\alpha \int_{u_0}^{u} (U_1/U'_1)\,du\right]. \tag{7.9}$$

In the same manner, three independent integrals of the equation in V, corresponding to (7.6), are

$$v_1 = 1, \quad v_2 = V_1, \quad v_3 = V_1 \int_{v_0}^{v} (V_2/V'_1)\,dv - \int_{v_0}^{v} (V_1 V_2/V'_1)\,dv, \tag{7.10}$$

in which

$$V_0 = V_1''/V_1 + \alpha V_1/V'_1, V_2 = \exp\left[\alpha \int_{u_0}^{v} (V_1/V'_1)\,dv\right]. \tag{7.11}$$

Now if the left member of (7.3) be denoted by $F(U)$

$$F(u_1) = -\alpha, \quad F(u_2) = 0, \quad F(u_3) = U_2 - \int_{u_0}^{u} U'_2 du = U_2(u_0).$$

We may suppose $U_2(u_0) = 1$. Similarly,

$$G(v_1) = \alpha, \quad G(v_2) = 0, \quad G(v_3) = 1.$$

The following combinations of u_i and v_i give the four solutions $y^{(i)}$

$$(7.12)\quad \begin{aligned} y^{(1)} &= u_1 + u_2 + \alpha u_3 + v_1 - \alpha v_3 = 2 + U_1 + \alpha(u_3 - v_3), \\ y^{(2)} &= u_1 + \alpha u_3 + v_1 + v_2 - \alpha v_3 = 2 + V_1 + \alpha(u_3 - v_3), \\ y^{(3)} &= u_2 + v_2 = U_1 + V_1, \\ y^{(4)} &= u_3 + v_3 = U_1 \int_{u_0}^{u} \frac{U_2}{U'_1}\, du - \int_{u_0}^{u} \frac{U_1 U_2}{U'_1}\, du \\ &\qquad + V_1 \int_{v_0}^{v} \frac{V_2}{V'_1}\, dv - \int_{v_0}^{v} \frac{V_1 V_2}{V'_1}\, dv, \end{aligned}$$

which are linearly independent as may be verified by computing the value of $| y_{vv}^{(i)}, y_u^{(i)}, y_v^{(i)}, y^{(i)} |$ when $\alpha = 0$.

The finite equations of the lines L_z and L_ρ may be found by substituting the expressions (7.12) in (1.2) and eliminating the functions U_1, V_1. The results are, respectively,

$$(7.13)\quad x_1 + \alpha x_4 = x_1 - x_2 + x_3 = 0, \quad x_2 - \alpha x_4 = x_1 - x_2 - x_3 = 0.$$

If $\alpha = 0$ the lines meet in the point $(0, 0, 0, 1)$. In the contrary case there is no point of intersection.

A TERNARY ANALOGUE OF ABELIAN GROUPS.

By D. H. LEHMER.*

Introduction. A class K with an operation called multiplication applied to *pairs* of elements of K is an abstract group provided certain postulates are satisfied. Unfortunately the name group does not suggest the *binary* character of multiplication. The entities with which this note is concerned are similar to groups, the class K being subjected to a *ternary* operation however. For want of a more descriptive name we have called them *triplexes.* The need for their consideration arose in an attempt to obtain solutions of a pair of functional equations, but their investigation would seem justified by intrinsic interest especially when compared with Abelian groups.

The system of postulates on which we base our investigation is modelled after Hurwitz's † system for Abelian groups, and accordingly differs somewhat from the definitions of a group found in most treatises on group theory. In this way we reduce the proofs of several theorems to a minimum and have at the same time a more perfect system from a strictly logical point of view. The role that Abelian groups play in this theory is described in § 3. The rest of the paper deals with finite triplexes and concepts analogous to the fundamental notions of the theory of Abelian groups such as order and inverse of an element, sub-group, cyclic group, quotient group, etc. Two notions however are conspicuous by their absence, the unit and the basis. Other facts such as the existence of triplexes with no subtriplex stand out as being different from what one might expect from the theory of Abelian groups.

1. *Definition of a Triplex.* A class K with an operation $\cdot$ between triplets of elements is called a triplex if the following postulates hold.

Postulate I.
$$\begin{aligned}(a\cdot b\cdot c)\cdot d\cdot e &= d\cdot(a\cdot b\cdot c)\cdot e = d\cdot e\cdot(a\cdot b\cdot c)\\ &= (a\cdot b\cdot d)\cdot c\cdot e = (a\cdot b\cdot e)\cdot c\cdot d = (a\cdot c\cdot d)\cdot b\cdot e\\ &= (a\cdot c\cdot e)\cdot b\cdot d = (a\cdot d\cdot e)\cdot b\cdot c = (b\cdot c\cdot d)\cdot a\cdot e\\ &= (b\cdot c\cdot e)\cdot a\cdot d = (b\cdot d\cdot e)\cdot a\cdot c = (c\cdot d\cdot e)\cdot a\cdot b\end{aligned}$$

provided $a, b, c, d, e,$ and all the expressions in which these letters are involved in the above, belong to K.

Postulate II. If $a, b, c,$ belong to $K,$ there is an element x of K such that $a\cdot b\cdot x = c.$

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† *Annals of Mathematics,* Ser. 2, Vol. 15, p. 93.

The number of elements in K is called the order of the triplex and is specified when necessary by adding one of the postulates:

Postulate III_1. K contains n elements.

Postulate III_2. K contains infinitely many elements.

According as III_1 or III_2 holds, the triplex is called finite or infinite. We proceed to deduce 5 fundamental theorems from Postulates *I* and *II*.

THEOREM 1. *If* a_1, a_2, a_3, *and one of the six expressions obtained from* $a_1 \cdot a_2 \cdot a_3$ *by permuting the letters, belong to* K, *then these six expressions are equal.*

Proof. Suppose for definiteness that $a_1 \cdot a_2 \cdot a_3$ belongs to K. Choose x_1, so that $a_2 \cdot a_3 \cdot x_1 = a_1$. In Postulate *I* set $a = d = a_2$, $b = e = a_3$, $c = x_1$. The first two equalities of Postulate *I* become

$$a_1 \cdot a_2 \cdot a_3 = a_2 \cdot a_1 \cdot a_3 = a_2 \cdot a_3 \cdot a_1.$$

Next choose x_2 so that $a_1 \cdot a_3 \cdot x_2 = a_2$. In Postulate *I* set $a = e = a_1$, $b = d = a_3$, $c = x_2$. Then we have, since $a_2 \cdot a_3 \cdot a_1$ is now in K

$$a_2 \cdot a_3 \cdot a_1 = a_3 \cdot a_2 \cdot a_1 = a_3 \cdot a_1 \cdot a_2.$$

Finally choose x_3 so that $a_1 \cdot a_2 \cdot x_3 = a_3$ and set $a = d = a_1$, $b = e = a_2$, $c = x_3$. Then since $a_3 \cdot a_1 \cdot a_2$ belongs to K, we have

$$a_3 \cdot a_1 \cdot a_2 = a_1 \cdot a_3 \cdot a_2 = a_1 \cdot a_2 \cdot a_3.$$

Hence the theorem.

Definition I. $a \cdot b \cdot c \cdot d \cdot e = (a \cdot b \cdot c) \cdot d \cdot e$.

THEOREM 2. *If the letters* a, b, c, d, e, *and their combinations appearing in Postulate I belong to* K, *then* $a \cdot b \cdot c \cdot d \cdot e$ *is unaltered by permuting the letters.*

Proof. Consider the 10 expressions obtained by striking out the expressions $d \cdot (a \cdot b \cdot c) \cdot e$ and $d \cdot e \cdot (a \cdot b \cdot c)$ from the list of Postulate *II*. They are all of the form

$$(a_1 \cdot a_2 \cdot a_3) \cdot a_4 \cdot a_5 = a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5.$$

In each of these 10 expressions we may, by Theorem 1, permute the letters in the parenthesis and interchange the two letters outside the parenthesis. We obtain 120 equal expressions no two of which involve the 5 letters exactly in the same order. Applying Definition *I*, we obtain all the $5!$ permutations of $a \cdot b \cdot c \cdot d \cdot e$. These expressions are equal; hence the theorem.

THEOREM 3. *If a, b, c belong to K, then $a \cdot b \cdot c$* belongs to K.

Proof. We may choose elements x, y, z, and w so that

$$a = b \cdot c \cdot x, \qquad x = a \cdot c \cdot y, \qquad a = y \cdot c \cdot z, \qquad z = a \cdot c \cdot w.$$

Then by Theorem 2

$$a = y \cdot c \cdot z = y \cdot c \cdot a \cdot c \cdot w = (a \cdot c \cdot y) \cdot w \cdot c = x \cdot w \cdot c.$$

But $z = a \cdot c \cdot w = b \cdot c \cdot x \cdot c \cdot w = (x \cdot w \cdot c) \cdot b \cdot c = a \cdot b \cdot c$. Thus $a \cdot b \cdot c$ is equal to an element z of K. Hence the theorem.

THEOREM 4. *The element x of Postulate II is unique.*

Proof. Suppose the theorem is false. It is then possible to find two distinct elements x_1 and x_2 of K so that

$$c = a \cdot b \cdot x_1 = a \cdot b \cdot x_2.$$

Let y be so chosen that $x_2 = a \cdot x_1 \cdot y$. Then

$$c = a \cdot b \cdot x_2 = a \cdot b \cdot a \cdot x_1 \cdot y = (a \cdot b \cdot x_1) \cdot a \cdot y = c \cdot a \cdot y.$$

Now let z be chosen so that $x_1 = a \cdot c \cdot z$. Then

$$x_1 = a \cdot z \cdot c = a \cdot z \cdot c \cdot a \cdot y = a \cdot x_1 \cdot y = x_2.$$

But this is impossible since x_1 and x_2 are distinct.

THEOREM 5. *If $a \cdot b \cdot x_1 = a \cdot b \cdot x_2$, then $x_1 = x_2$.*

Proof. This is in fact another statement of Theorem 4.

2. *Associated Elements.*

THEOREM 6. *If a and b are any elements of K, then the solution x of $a \cdot b \cdot x = b$ depends on a alone.*

Proof. Let c be an element of K. Then since $a \cdot b \cdot x = b$ we have

$$a \cdot (a \cdot b \cdot x) \cdot c = a \cdot b \cdot c = a \cdot b \cdot (a \cdot c \cdot x).$$

Hence by Theorem 5, $c = a \cdot c \cdot x$. Thus x does not depend on b.

We write $x = a'$ and call a' the element associated with a. By Theorem 4, a' is unique. Obviously $(a')' = a$. Also if a and a' both occur in an expression with $\cdot$ they may be cancelled.

THEOREM 7. $a' \cdot b' \cdot c' = (a \cdot b \cdot c)'$.

Proof. Let $a' \cdot b' \cdot c' = x_1$, and $(a \cdot b \cdot c)' = x_2$ and let d be any element

in K. Then $a \cdot b \cdot c \cdot d \cdot x_1 = d = (a \cdot b \cdot c) \cdot d \cdot x_2$. Hence by Theorem 5, $x_1 = x_2$, which is the theorem.

3. *Abelian Groups.* Given a class K and an operation $\cdot$, applied to pairs of elements under which K is an Abelian group, then it follows that K is also a triplex under the operation $\cdot$ applied to triplets of elements. In fact such a class satisfies Postulates *I* and *II*. In such a case the triplex is called the extension of the group. Obviously every Abelian group has only one extension. However some triplexes are the extensions of more than one Abelian group. For example we may take for K the class of all positive and negative rational numbers (0 excepted). The two groups under the two operations

$$a \cdot b = ab \qquad \text{and} \qquad a \cdot b = -ab$$

when extended give the same triplex since in both cases $a \cdot b \cdot c = abc$. It is possible to choose a class K and an operation $\cdot$ which when applied to pairs of elements does not give an Abelian group but nevertheless when $\cdot$ is applied to triplets we obtain a triplex. Consider for example the class K of all positive and negative odd integers. Under the operation $a \cdot b = a + b$, the class K is not a group since there is no closure property. Under the operation $a \cdot b \cdot c = a + b + c$ however, K is a triplex. Some triplexes are not the extension of any Abelian group. The simplest example of such a triplex is the following. Let the class K consist of two elements a and b. The operation $\cdot$ is defined by the following table.

$$a \cdot a \cdot a = b \qquad a \cdot a \cdot b = a \cdot b \cdot a = b \cdot a \cdot a = a$$
$$b \cdot b \cdot b = a \qquad a \cdot b \cdot b = b \cdot a \cdot b = b \cdot b \cdot a = b.$$

Then K is a triplex under $\cdot$. But it is impossible to interpret $a \cdot a$, $a \cdot b$, and $b \cdot b$ in a consistent way. To give a concrete example of such a triplex set $a = i$, $b = -i$ and $a \cdot b \cdot c = abc$.

THEOREM 8. *If a triplex contains an element ϵ equal to its own associate ϵ', then the triplex is an extension of an Abelian group of which ϵ is the unit.*

Proof. For every pair (a, b) of elements of K there is a unique third element defined by $a \cdot b \cdot \epsilon = c$. We may express this fact by writing $a \cdot b = c$ and this notation is consistent because

$$(a \cdot b) \cdot d = (a \cdot b \cdot \epsilon) \cdot d \cdot \epsilon = a \cdot b \cdot d \cdot \epsilon \cdot \epsilon' = a \cdot b \cdot d.$$

Applying $\cdot$ to pairs of elements of K, we obtain an Abelian group of which as we have just seen, the triplex is an extension. Moreover

$$a \cdot \epsilon = a \cdot \epsilon \cdot \epsilon = a \cdot \epsilon \cdot \epsilon' = a.$$

Hence ϵ is the unit of the group. This proves the theorem.

THEOREM 9. *If a triplex is the extension of exactly r distinct Abelian groups, then the class K contains exactly r distinct elements* $\epsilon_1, \epsilon_2, \cdots, \epsilon_r$ *such that each is equal to its associate. In fact these elements are the units of the r groups.*

Proof. Let G_ν be the ν-th group $(\nu = 1, 2, \cdots, r)$ and let ϵ_ν be its unit. Let us symbolize its operation by $(ab)_\nu$. Then since $a \cdot b \cdot c = (a(bc)_\nu)_\nu$ we have for $b = c = \epsilon_\nu$

$$a \cdot \epsilon_\nu \cdot \epsilon_\nu = (a(\epsilon_\nu \epsilon_\nu)_\nu)_\nu = (a\epsilon_\nu)_\nu = a$$

But $a \cdot \epsilon_\nu \cdot \epsilon'_\nu = a$, hence by Theorem 5, $\epsilon_\nu = \epsilon'_\nu$. Suppose that these ϵ's are not all distinct, so that $\epsilon_\lambda = \epsilon_\mu$, then

$$(ab)_\lambda = a \cdot b \cdot \epsilon_\lambda = ((ab)_\lambda \epsilon_\lambda)_\lambda = ((ab)_\mu \epsilon_\lambda)_\mu = (ab)_\mu.$$

But this is impossible since a and b are quite arbitrary, and G_λ and G_μ are distinct. Hence the ϵ's are distinct. Since a group contains only one unit it follows from Theorem 8 that our triplex contains not more than r distinct elements each equal to its own associate.

4. *Finite Triplexes.* In what follows we suppose that Postulate III_1 holds so that the triplex is of a finite order n.

THEOREM 10. *If the order n of a triplex is odd then the triplex is an extension of an Abelian group.*

Proof. With every element a of K we can associate a'. Since n is odd at least one element is its own associate. Hence the theorem follows from Theorem 8.

5. *The Degree.* Consider any element a and its "powers."

$$a, \quad a \cdot a \cdot a = a^3, \quad a^3 \cdot a \cdot a = a^5, \cdots. \tag{1}$$

Since the order of the triplex is finite this sequence of powers of a contains a finite number of distinct elements. Let

$$a^{2k+1} = a^{2l+1} = a^{2m+1}, \qquad (k < l < m).$$

Then we have

$$a \cdot a \cdot a^{2k+1} = a^{2l+1} \cdot a^{2(l-k)+1} \cdot a = a \cdot a \cdot a^{2l+1}.$$

Hence by Theorem 5,

$$a = a^{2(l-k)+1} = a^{2(m-k)+1}.$$

Hence the sequence (1) contains two powers of a equal to a. If

$$a = a^{2p+1} = a^{2q+1},$$

then it follows in the same way that $a = a^{2(p\pm q)+1}$. Hence the set of all numbers r for which $a^{2r+1} = a$ is a module and coincides with all integer multiples of an integer α. Hence we have

THEOREM 11. *For each element a there exists an integer α such that $a^{2k+1} = a^{2l+1}$ if and only if $k \equiv l \pmod{\alpha}$.*

The number α is called the degree of a.

THEOREM 12. *$a' = a^{2\alpha-1}$ where α is the degree of a.*

Proof. Let b be an arbitrary element and consider the expression

$$a \cdot a \cdot (a \cdot b \cdot a^{2\alpha-1}) = a \cdot b \cdot a^{2\alpha+1} = a \cdot a \cdot b.$$

Hence $b = a \cdot b \cdot a^{2\alpha-1}$, and $a^{2\alpha-1}$ is the unique element associated with a. Moreover a and a' are of the same degree.

6. *Sub-Triplex.* Let T be a triplex defined by a class K and an operation $\cdot$. If there is a triplex T_1 defined by a sub-class of K and the same operation $\cdot$, T_1 is called a sub-triplex of T. For example, the elements $a, a^3, a^5, \cdots, a^{2\alpha-1}$ form, under $\cdot$, a sub-triplex, provided, of course, $\alpha < n$.

Let $u_1, u_2, u_3, \cdots, u_r$, be any sub-triplex T_1 of T. Consider u_m and the following sequences

$$(2) \qquad a \cdot u_m \cdot u_1, \quad a \cdot u_m \cdot u_2, \quad a \cdot u_m \cdot u_3, \cdots, a \cdot u_m \cdot u_r$$

$$(3) \qquad b \cdot u_m \cdot u_1, \quad b \cdot u_m \cdot u_2, \quad b \cdot u_m \cdot u_3, \cdots, b \cdot u_m \cdot u_r$$

where a and b belong to K. If any term of (2) is equal to any term of (3), then (2) and (3) are identical except for order. To show this suppose that $a \cdot u_m \cdot u_i = b \cdot u_m \cdot u_j$. Now u_j has an associate among the u's since they form a triplex. Hence

$$a \cdot u_m \cdot (u_i \cdot u'_j \cdot u_k) = b \cdot u_m \cdot u_j \cdot u'_j \cdot u_k = b \cdot u_m \cdot u_k.$$

Let k run from 1 to r. Then by Postulate *II* and Theorem 5 the expression $(u_i \cdot u'_j \cdot u_k)$ runs over all the u's without repetition. Hence (2) is some permutation of (3).

In (2) let a run over all the elements of K. From the resulting sequences let us save only those that are distinct and reject one of any pair of sequences which differ only in order. We are left then with a table of the form

$$(4) \qquad \begin{array}{cccccc} a_1 \cdot u_m \cdot u_1 & a_1 \cdot u_m \cdot u_2 & \cdot & \cdot & \cdot & a_1 \cdot u_m \cdot u_r \\ a_2 \cdot u_m \cdot u_1 & a_2 \cdot u_m \cdot u_2 & \cdot & \cdot & \cdot & a_2 \cdot u_m \cdot u_r \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ a_s \cdot u_m \cdot u_1 & a_s \cdot u_m \cdot u_2 & \cdot & \cdot & \cdot & a_s \cdot u_m \cdot u_r \end{array}$$

whose elements are distinct. In fact two equal elements could not be in different lines by the above reasoning, nor could they be in the same line since this would force two u's to be equal by Theorem 5. Let b be any element of K, then there is an element a for which

$$b = a \cdot u_m \cdot u_1.$$

Either a is one of the a_i $(i = 1, 2, \cdots, s)$ chosen above or else it was discarded in forming the table (4) because b had already occurred in the table. In either case b occurs in (4) and only once, since the elements of (4) are distinct. In short the elements of (4) are the elements of K. We have then

THEOREM 13. *The order of a sub-triplex is a proper divisor of the order of the triplex.*

As an immediate consequence of this theorem we have

COROLLARY 1. *The degree of any element of a triplex is a divisor of the order of the triplex.*

COROLLARY 2. *If a is any element of a triplex of order n, then $a^{2n+1} = a$.*

7. *The Quotient Triplex.* Let us return to the table (4) and discuss more of its properties.

THEOREM 14. *Let i, j, and k be any integers $\leqq s = n/r$. Then the set of r elements*

$$a_i \cdot a_j \cdot a_k \cdot u_m \cdot u_t \qquad (t = 1, 2, \cdots, r) \tag{5}$$

is, in some order, a line of the table (4).

Proof. Write $a = a_i \cdot a_j \cdot a_k$. Suppose $a \cdot u_m \cdot u_1$ lies in the ν-th line of (4). Then we can write

$$a \cdot u_m \cdot u_1 = a_\nu \cdot u_m \cdot u_\lambda \qquad (1 \leq \lambda \leq r).$$

If now u_ρ is any u, the corresponding member of (5) can be written

$$\begin{aligned} a \cdot u_m \cdot u_\rho &= a \cdot u_m \cdot (u_\rho \cdot u'_1 \cdot u_1) = (a \cdot u_m \cdot u_1) \cdot u_\rho \cdot u'_1 \\ &= a_\nu \cdot u_m \cdot (u_\lambda \cdot u_\rho \cdot u'_1) = a_\nu \cdot u_m \cdot u_\sigma \end{aligned}$$

hence $a \cdot u_m \cdot u_\rho$ is also in the ν-th line of the table and the theorem follows at once.

THEOREM 15. *If all the elements of any line of the table* (4) *be replaced by their associates, the result is a permutation of the elements of one of the lines of the table.*

Proof. Let us replace the elements of the ν-th line by their associates. Let the first element $a_\nu \cdot u_m \cdot u_1$ have its associate in the λ-th line so that $(a_\nu \cdot u_m \cdot u_1)' = a_\lambda \cdot u_m \cdot u_\omega$. Then the associate of any element in the ν-th line can be written

$$\begin{aligned}(a_\nu \cdot u_m \cdot u_\mu)' &= a'_\nu \cdot u'_m \cdot u'_\mu = a'_\nu \cdot u'_m \cdot u'_\mu \cdot u'_1 \cdot u_1 \\ &= (a_\nu \cdot u_m \cdot u_1)' \cdot u'_\mu \cdot u_1 = a_\lambda \cdot u_m \cdot (u_\omega \cdot u'_\mu \cdot u_1) \\ &= a_\lambda \cdot u_m \cdot u_\tau.\end{aligned}$$

Hence the associate lies also in the λ-th line. From this the theorem follows.

It becomes convenient to introduce a notation for the lines of the table (4). Accordingly we shall mean by l_i the i-th line or any permutation of its elements. We also define the result of operating by $\cdot$ on triplets of lines as follows

$$l_i \cdot l_j \cdot l_k = l_m,$$

where l_m is the line in which the element $a_i \cdot a_j \cdot a_k$ occurs. We further designate by l'_i the line containing the associates of the elements of l_i, and call either line the associate of the other. To show that this notation is proper we prove

THEOREM 16. *If l_k is any line of* (4), *then* $l_i \cdot l'_i \cdot l_k = l_k$.

Proof. Let $l'_i = l_j$. Then $l_i \cdot l'_i \cdot l_k \cdot l_i \cdot l_j$ is the line in which $a_i \cdot a_j \cdot a_k$ occurs. But a_j is the associate of some element of l_i. Hence we may write

$$a_j = (a_i \cdot u_m \cdot u_\rho)' = a'_i \cdot u'_m \cdot u'_\rho.$$

Therefore the line in question is that in which the element

$$a_i \cdot a'_i \cdot u'_m \cdot u'_\rho \cdot a_k = a_k \cdot u'_m \cdot u'_\rho \cdot u_m \cdot u'_m = a_k \cdot u_m \cdot (u'_m \cdot u'_m \cdot u'_\rho)$$

occurs. But $u'_m \cdot u'_m \cdot u'_\rho$ is some u, so that this element occurs in the k-th line and no other. Hence the theorem. We are now able to prove

THEOREM 17. *Under the operation $\cdot$ defined above the set of l's is a triplex.*

Proof. It is obvious that Postulate *I* is satisfied. As for Postulate *II* consider l_i, l_j and l_k as given. Then the expression $l_i' \cdot l_j' \cdot l_k = l_x$ is such that

$$l_i \cdot l_j \cdot l_x = l_i \cdot l'_i \cdot (l_j \cdot l'_j \cdot l_k) = l_j \cdot l'_j \cdot l_k = l_k.$$

Hence Postulate II is satisfied and the theorem follows.

It is clear that this triplex of lines is actually independent of the choice of a_i $(i = 1, 2, 3, \cdots, s)$. In fact some other choice would only permute

the elements of some of the lines. Furthermore the l's themselves do not actually depend on u_m for the same reason. Hence this triplex is uniquely determined by T and T_1 and corresponds to the quotient group G/H of G and its sub-group H.

THEOREM 18. *The quotient triplex is an extension of an Abelian group.*

Proof. The element u_1 must occur in the table (4). In fact there is a line l_μ which is made up of all the r elements of the sub-triplex T_1. Moreover a_μ is an element of T_1 as is also $a_\mu{}^3$. Hence $l_\mu \cdot l_\mu \cdot l_\mu = l_\mu$, so that $l'_\mu = l_\mu$, and by Theorem 8 the quotient triplex is an extension of an Abelian group.

8. *Cyclic Triplexes.* We proceed to give a few theorems concerning triplexes of a simple type. A triplex which consists of the distinct (odd) powers of a single element is called cyclic. In other words, a triplex is cyclic if and only if it contains an element whose degree is the order of the triplex. Such a triplex is simply isomorphic with the set of distinct odd powers of a primitive n-th root of unity, with the operation $\cdot$ as multiplication, or the set of odd integers $\leqq n$ under addition modulo n. From either of these points of view the following theorems are easily established.

THEOREM 19. *The order of any sub-triplex of a cyclic triplex contains exactly the same power of 2 as does the order of the triplex.*

THEOREM 20. *The sub-triplexes of a cyclic triplex are cyclic.*

THEOREM 21. *All cyclic triplexes of a fixed order n are simply isomorphic.*

THEOREM 22. *If n is the degree of an element a of a cyclic triplex of order n, then the degree of $b = a^\omega$ (ω odd) is $n/(n, \omega)$.*

THEOREM 23. *A cyclic triplex of order n contains exactly $\phi(2n)$ elements of degree n.*

Here ϕ is the totient function. More generally we have

THEOREM 24. *If $n = d\,\delta$ is any factorization of the order n of a cyclic triplex in which δ is odd, then there are exactly $\phi(2d)$ elements of the triplex of degree d.*

THEOREM 25. *No cyclic triplex of even order is an extension of a group.*

Proof. In fact the unit element of such a group would constitute a sub-triplex of order 1. This is impossible by Theorem 19.

THEOREM 26. *If the order of a cyclic triplex is a power of 2, then the triplex has no sub-triplexes.*

Proof. The existence of such a sub-triplex would contradict Theorem 19. As a converse of this theorem we have

THEOREM 27. *If a triplex T has no sub-triplex, then T is cyclic, and its order n is a power of 2.*

Proof. The degree of any element of T cannot be less than n, since otherwise we would have a sub-triplex of T. Hence T is cyclic. By Theorem 22 it follows that n must be prime to all odd numbers. That is, n is a power of 2.

9. *Examples.* We conclude with two simple examples of triplexes. The complete multiplication table of a triplex of order n is a cubical matrix of n^3 elements. On account of the symmetry of $a \cdot b \cdot c$ however, only one-sixth of this matrix need be given. But it is more convenient to form a plane table as follows

Example 1:

$$a = a \cdot a \cdot b = a \cdot c \cdot d = b \cdot b \cdot b \cdot = b \cdot c \cdot c = b \cdot d \cdot d$$
$$b = a \cdot a \cdot a \cdot = a \cdot b \cdot b = a \cdot c \cdot c = b \cdot c \cdot d = a \cdot d \cdot d$$
$$c = a \cdot a \cdot d = a \cdot b \cdot c = b \cdot b \cdot d = c \cdot c \cdot d = d \cdot d \cdot d$$
$$d = a \cdot a \cdot c = a \cdot b \cdot d = b \cdot b \cdot c = c \cdot c \cdot c = c \cdot d \cdot d.$$

This triplex may also be described by

$$a^5 = a, \qquad c^5 = c, \qquad a \cdot a \cdot c = c \cdot c \cdot c.$$

Each letter is of degree 2 and $a' = b$, and $c' = d$. This triplex contains two cyclic sub-triplexes namely (a, b) and (c, d) both of which are simply isomorphic with the two letter triplex of § 3. The quotient triplex of each is the extension of two Abelian groups and may be described by

$$l_1 = l_1 \cdot l_1 \cdot l_1 = l_1 \cdot l_2 \cdot l_2$$
$$l_2 = l_1 \cdot l_1 \cdot l_2 = l_2 \cdot l_2 \cdot l_2.$$

Example 2:

$a^{13} = a$. This is a cyclic triplex of 6 letters. It contains only one sub-triplex namely (a^3, a^9). The quotient triplex is described by

$$l'_1 = l_1, \qquad l'_2 = l_3, \qquad l_2^5 = l_2.$$

STANFORD UNIVERSITY.

ON THE ASYMPTOTIC REPARTITION OF THE VALUES OF REAL ALMOST PERIODIC FUNCTIONS.

By AUREL WINTNER.

In a previous paper * upon bounded matrices representing,† in the sense of Frobenius, the groups ‡ corresponding to the almost periodic functions of Bohr ¶ there is introduced the repartition function ("inverse function in the large") of a real almost periodic function. The presentation of the repartition function there given depends to some extent upon the theory of almost periodic matrices set forth in that paper. The aim of the present note is a more direct treatment which avoids any connection with the theory of matrices.—The methods are also valid for various generalizations ‖ of the notion of the almost-periodicity which are, however, for simplicity not treated in the present note.

Let f denote a real almost periodic function of the real argument t and m the largest, M the smallest number for which

$$m \leqq f(t) \leqq M, \qquad -\infty < t < +\infty. \tag{1}$$

Let

$$[f \leqq \xi]_T$$

denote the set of the points t for which

$$[f \leqq \xi]_T: \qquad -T \leqq t \leqq T \quad \text{and} \quad f(t) \leqq \xi \tag{2}$$

where ξ is any real number and T any positive number. The function **

* A. Wintner, "Diophantische Approximationen und Hermitesche Matrizen," *Mathematische Zeitschrift*, Vol. 30 (1929), pp. 290-319.

† These ideas have been inaugurated by O. Toeplitz, "Theorie der L-Formen," *Mathematische Annalen*, Vol. 70 (1911), p. 353; "Ueber das Wachstum der Potenzreihen in ihrem Konvergenzkreise. I," *Mathematische Zeitschrift*, Vol. 12 (1922), p. 195, and a not yet published paper on ordinary Dirichlet series.

‡ The abstract groups represented by the matrices which correspond to the limiting periodic (grenzperiodisch) subclass § of the class of the almost periodic functions have been studied recently by H. Schwerdtfeger, "Fastzyklische Gruppen," *Goettinger Nachrichten*, 1931, pp. 43-48.

§ H. Bohr, "Grenzperiodische Funktionen," *Acta Mathematica*, Vol. 52 (1928), pp. 127-133. Cf. G. D. Birkhoff, "On the Periodic Motions of Dynamical Systems," *Acta Mathematica*, Vol. 50 (1927), pp. 395-397.

¶ H. Bohr, "Zur Theorie der fast-periodischen Funktionen. I," *Acta Mathematica*, Vol. 45 (1925), pp. 29-127.

‖ Cf. H. Bohr, "Ueber die Verallgemeinerungen fastperiodischer Funktionen," *Mathematische Annalen*, Vol. 100 (1928), pp. 357-366.

** We denote by mes P the measure of the point set P.

$$\rho_T(\xi) = (1/2T) \cdot \text{mes}\,[f \leqq \xi]_T \tag{3}$$

can be interpreted as yielding the relative frequency of the inequality $f(t) \leqq \xi$ in the range $-T \leqq t \leqq T$. In other words the difference

$$\rho_T(\eta) - \rho_T(\xi) \quad \text{where} \quad \eta > \xi \tag{4}$$

represents the geometric probability that the number f lies in the strip $\xi < f \leqq \eta$ provided $-T \leqq t \leqq T$. The difference (4) is, according to (2) and (3), never negative, i. e. $\rho_T(\xi)$ is, for any fixed value of T, a monotone function of ξ for which we have, according to (1), (2), and (3),

$$\rho_T(\xi) \equiv 0 \text{ for } -\infty < \xi < m_T \text{ and } \rho_T(\xi) \equiv 1 \text{ for } M_T < \xi < +\infty \tag{5}$$

where

$$\begin{aligned} m_T &= \min_{-T \leqq t \leqq T} f(t) \geqq \lim_{T=+\infty} m_T = m, \\ M_T &= \max_{-T \leqq t \leqq T} f(t) \leqq \lim_{T=+\infty} M_T = M. \end{aligned} \tag{6}$$

Since $f(t)$ is almost periodic and therefore continuous it follows from the definitions (1), (2), and (3) that the interval $m_T \leqq \xi \leqq M_T$ contains no subinterval in which $\rho_T(\xi)$ would be constant (this statement is simply the translation of the theorems of Bolzano and Weierstrass in the language of the repartition functions ρ). On the other hand the function $\rho_T(\xi)$ may have some discontinuity points (which may even lie everywhere dense in the interval $m_T \leqq \xi \leqq M_T$); this is, for instance, the case if the almost periodic function $f(t)$ is constant in some subintervals of the interval $-T \leqq t \leqq T$.— Since $f(t)$ is continuous it is also integrable in the sense of Lebesgue in any finite interval $-T \leqq t \leqq T$; the function (3) of ξ is obviously nothing more than the measure function (Massfunktion) of Lebesgue, belonging to the function $f(t)$ in the interval $-T \leqq t \leqq T$. It therefore follows, according to the theorems of Lebesgue * concerning the connection of his integral notion with that of Stieltjes, that

$$(1/2T) \int_{-T}^{T} (f(t))^n \, dt = \int_{-\infty}^{+\infty} \xi^n \, d\rho_T(\xi); \qquad (n = 0, 1, 2, \cdots), \tag{7}$$

i. e.

$$(1/2T) \int_{-T}^{T} (z - f(t))^{-1} \, dt = \int_{-\infty}^{+\infty} (z - \xi)^{-1} \, d\rho_T(\xi) \tag{8}$$

where z any constant (real or complex) which does not lie in the real segment ("Schlitz") $m_T \leqq z \leqq M_T$.

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* H. Lebesgue, "Sur l'intégrale de Stieltjes et sur les operations functionelles linéaires," *Comptes Rendus*, Vol. 150 (1910), pp. 86-88.

The results of the paper mentioned in the introduction which are independent of the theory of matrices concern an analogous *repartition theory belonging to the full definition domain* $-\infty < t < +\infty$ of the almost periodic function $f(t)$, i. e. *the (asymptotic) behaviour of the relative frequencies or probabilities in the large:* $T = +\infty$. We shall write

$$\mu(g) = \lim_{T=+\infty} (1/2T) \int_{-T}^{T} g(t)\,dt \tag{9}$$

provided this Hadamard average exists which is certainly the case if g is almost periodic and therefore if $g = f^n$ where $n = 0, 1, 2, \cdots$.—The results are as follows:

I. *There exists a monotone non-decreasing function* $\rho(\xi)$ *for which*

$$\int_{-\infty}^{+\infty} \xi^n \, d\rho(\xi) = \mu(f^n); \qquad (n = 0, 1, 2, \cdots) \tag{10}$$

and the solution $\rho(\xi)$ *of* (10) *is uniquely determined if one normalizes it, for instance, by placing*

$$\rho(-\infty) = 0; \qquad \rho(\xi) = \rho(\xi + 0), \qquad -\infty < \xi < +\infty; \tag{11}$$

i. e. the Hamburger momentum problem belonging to the sequences $\{\mu(f^n)\}$ is solvable and well-determined (déterminé).—This uniquely determined function $\rho(\xi)$ obviously possesses the usual statistical properties of a repartition function; from (10) and (9) there follows, for instance,

$$\mu(f^0) = \mu(1) = 1 = \int_{-\infty}^{+\infty} d\rho(\xi),$$

i. e. the total probability is equal to unity.

II. *The monotone function* $\rho(\xi)$ *which is uniquely determined by its formal properties* (10) *and* (11) *is effectively the asymptotic repartition function of the almost periodic function* $f(t)$. *In a more precise manner we have*

$$\rho(\xi) = \lim_{T=+\infty} (1/2T) \cdot \text{mes}\,[f \leqq \xi]_T, \quad \text{i. e.} \quad \rho(\xi) = \lim_{T=+\infty} \rho_T(\xi) \tag{12}$$

for all continuity points of the function $\rho(\xi)$ which is monotone and has therefore at most a countable set of discontinuity points.—From (5), (6), and (12) there obviously follows

$$\rho(\xi) \equiv 0 \text{ for } -\infty < \xi < m \text{ and } \rho(\xi) \equiv 1 \text{ for } M < \xi < +\infty. \tag{13}$$

III. *The function* $\rho(\xi)$ (which may have discontinuity points) *cannot*

be constant in an interval $\alpha < \xi < \beta$ *if* $m < \alpha$ *or* $\beta < M$ [cf. (1)]. This is the asymptotic refinement of the Bolzano-Weierstrass theorems mentioned above.

In the special case where the almost periodic function $f(t)$ is *periodic* and has the period τ it is clear that

$$\rho_\tau(\xi) = \lim_{T=+\infty} \rho_T(\xi) \equiv \rho(\xi) \tag{12'}$$

for *all* ξ, where T denotes as before a continuous parameter. In other words in the special case of periodic functions the limit (12) exists even if ξ is a discontinuity point of the asymptotic repartition function $\rho(\xi)$. In the paper mentioned in the introduction I was not able to show that the limit (12) also exists for the discontinuity points of the asymptotic repartition function $\rho(\xi)$ of any *almost* periodic function. Since that time it has been shown by Bohr * that not the apparatus applied by myself was at fault but that the limit (12) does not necessarily exist for an *almost* periodic function in the discontinuity points of its asymptotic repartition function. Since the theory of the Stieltjes transforms used by myself is accordingly the natural method † the present note applies, without modifications, for the demonstration of II the same apparatus.—One can start with the following theorem which follows readily from classical general results of Stieltjes and Helly ‡:

Let $\sigma_x(\xi)$ be a sheaf of functions of the variable ξ defined for $-\infty < \xi < +\infty$ and for all positive values of the parameter x of the sheaf. Suppose that $\sigma_x(\xi)$ is, for any fixed value of x, a monotone function of ξ:

$$\sigma_x(\eta) - \sigma_x(\xi) \geqq 0 \quad \text{for} \quad \eta > \xi \tag{14a}$$

and that there exist two constants, c and C, independent of x, for which

$$\sigma_x(\xi) \equiv 0 \text{ for } -\infty < \xi < c \text{ and } \sigma_x(\xi) \equiv 1 \text{ for } C < \xi < +\infty; \tag{14b}$$

suppose finally that there exists a domain § in the plane of the complex variable z in such a manner that the limit

$$\lim_{x=+\infty} \int_{-\infty}^{+\infty} (z - \xi)^{-1} \, d\sigma_x(\xi) \tag{14c}$$

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* H. Bohr, "Kleinere Beitraege zur Theorie der fastperiodischen Funktionen. II," *Det Kgl. Danske Videnskabernes Selskab. Meddelelser*, Vol. 10, Nr. 10, 1930.

† It is known that the Stieltjes-Helly theory is never able to yield any information at the discontinuity points of the limit function.

‡ Cf., for instance, A. Wintner, *Spektraltheorie der unendlichen Matrizen*, Leipzig, 1929, Chap. II.

§ By a domain we understand for simplicity a two-dimensional domain which does not contain any point of the real segment $c \leqq z \leqq C$ so that the integral behind the sign lim in (14c) obviously has a sense; cf. (14b).

exists for all points z of this domain. Then there exists in the range $-\infty < \xi < +\infty$ one and only one function $\sigma(\xi)$ for which

$$\sigma(\xi) = \sigma(\xi + 0) \tag{14\alpha}$$

for all values of ξ and

$$\sigma(\xi) = \lim_{x=+\infty} \sigma_x(\xi) \tag{14\beta}$$

for all continuity points of this function $\sigma(\xi)$ which is monotone (and has therefore at most a countable set of discontinuity points); furthermore, the function $\sigma(\xi)$, uniquely determined by (14α) and (14β), has the property that

$$\lim_{x=+\infty} \int_{-\infty}^{+\infty} (z-\xi)^{-1} d\sigma_x(\xi) = \int_{-\infty}^{+\infty} (z-\xi)^{-1} d\sigma(\xi) \tag{14\gamma}$$

for all values of z which do not lie in the segment $c \leqq z \leqq C$; finally we have [cf. (14b), (14β)]

$$\sigma(\xi) \equiv 0 \text{ for } -\infty < \xi < c \text{ and } \sigma(\xi) \equiv 1 \text{ for } C < \xi < +\infty \tag{14\delta}$$

so that

$$\int_{-\infty}^{+\infty} = \int_{c-0}^{C+0} \tag{14\epsilon}$$

in all occurring Stieltjes integrals.

Now if we put

$$c = m, \qquad C = M, \qquad x = T, \qquad \sigma_x = \rho_T, \qquad \sigma = \rho \tag{15}$$

then condition (14a) is fulfilled inasmuch as the difference (4) is never negative. Condition (14b) is satisfied according to (5) and (6). Finally if z denotes any number which does not lie in the segment $m \leqq z \leqq M$ of the z-plane then $g(t) = (z - f(t))^{-1}$ is, according to (1), an almost periodic function so that expression (8) approaches for $T = +\infty$ the Hadamard average (9):

$$\lim_{T=+\infty} \int_{-\infty}^{+\infty} (z-\xi)^{-1}\, d\rho_T(\xi) = \mu((z-f)^{-1}). \tag{15c}$$

All three conditions (14a), (14b), (14c) are accordingly satisfied.—The proof of I, II, and III (p. 341) proceeds now as follows.

Proof of I. On denoting by $\rho(\xi)$ the function $\sigma(\xi)$ which is uniquely defined by (14α) and (14β) equations (14γ), (15c), and (15) yield

$$\int_{-\infty}^{+\infty} (z-\xi)^{-1}\, d\rho(\xi) = \mu((z-f)^{-1}) \tag{16}$$

for all values of z which do not lie in the segment $m \leqq z \leqq M$ and therefore certainly for all values of z which do not lie in the circle $|z| \leqq R$ where

$$R \geqq |m|, \qquad R \geqq |M|. \tag{17}$$

Since the series

$$(z - f(t))^{-1} = \sum_{n=0}^{\infty} z^{-n-1}(f(t))^n$$

is, according to (1) and (17), uniformly convergent with respect to t for any fixed value of z for which $|z| > R$, we have

$$\mu((z-f)^{-1}) = \sum_{n=0}^{\infty} z^{-n-1}\mu(f^n) \quad \text{for} \quad |z| > R. \tag{18}$$

From (14ϵ), (15), and (17) there follows in an analogous manner

$$\int_{-\infty}^{+\infty} (z-\xi)^{-1}\, d\rho(\xi) = \sum_{n=0}^{\infty} z^{-n-1} \int_{-\infty}^{+\infty} \xi^n\, d\rho(\xi) \quad \text{for} \quad |z| > R. \tag{19}$$

On introducing (18) and (19) in (16) and comparing the coefficients one obtains the momentum equations (10) the solution $\rho(\xi) = \sigma(\xi)$ of which fulfills, according to (14α) and (14δ), the conditions (11). In order to finish the proof of I we show that the solution $\rho(\xi)$ found before is unique. The solution $\rho(\xi)$ of (10) and (11) is, according to Carleman,* certainly unique if

$$\sum_{n=0}^{\infty} \left[\int_{-\infty}^{+\infty} \xi^{2n}\, d\rho(\xi) \right]^{-1/2n} = +\infty. \tag{20}$$

Equation (20) is an obvious consequence of (14ϵ). In fact, we have for any number r which is larger than $|c|$ and $|C|$

$$0 < \int_{-\infty}^{+\infty} \xi^{2n}\, d\rho(\xi) = \int_{-r}^{r} \xi^{2n}\, d\rho(\xi) < r^{2n} \int_{-r}^{r} d\rho(\xi) = r^{2n}$$

which clearly is sufficient for (20).

Proof of II. Equation (12) and (13) follow, by virtue of (15), from (14β) and (14δ), respectively.

Proof of III. The function $f(t)$ is, by supposition, almost periodic,

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* T. Carleman, "Sur les séries asymptotiques," *Comptes Rendus*, Vol. 174 (1922), pp. 1527-1530, and "Sur les functions quasi-analytiques," *Wissenschaftliche Vortraege, gehalten am 5. Kongress der Skandinavischen Mathematiker*, Helsingfors, 1923, pp. 186-187.

i. e. it is continuous in every point of the range $-\infty < t < +\infty$ and has the property that there exists, for any fixed $\epsilon > 0$, a sequence

$$\cdots, \quad l_{-2}(\epsilon), \quad l_{-1}(\epsilon), \quad l_0(\epsilon), \quad l_1(\epsilon), \quad l_2(\epsilon), \cdots$$

of numbers $l_\nu = l_\nu(\epsilon)$ and a number $K = K(\epsilon)$, independent of ν, for which

$$0 < l_{\nu+1}(\epsilon) - l_\nu(\epsilon) < K(\epsilon), \qquad (\nu = 0, \pm 1, \pm 2, \cdots) \tag{21}$$

$$\lim_{\nu=+\infty} l_\nu(\epsilon) = +\infty, \qquad \lim_{\nu=-\infty} l_\nu(\epsilon) = -\infty$$

and

$$|f(t + l_\nu(\epsilon)) - f(t)| < \epsilon \quad \text{for} \quad -\infty < t < +\infty, \quad (\nu = 0, \pm 1, \pm 2, \cdots). \tag{22}$$

Let $\xi = \alpha$ be any *inner* point of the range $m \leqq \xi \leqq M$ of $f(t)$. Then there exists a number t_0 for which

$$f(t_0) = \alpha. \tag{23}$$

After α and t_0 are fixed one can determine, by virtue of the continuity of $f(t)$, for any sufficiently small $\delta > 0$ a number $\tau = \tau(\delta) > 0$ for which

$$|f(t) - \alpha| < \delta \quad \text{if} \quad t_0 - \tau(\delta) < t < t_0 + \tau(\delta), \tag{24}$$

inasmuch as α is an inner point $(m < \alpha < M)$. Conditions (22), (23), and (24) yield

$$|f(t + l_\nu(\delta)) - f(t_0)| < 2\delta \quad \text{for} \quad t_0 - \tau(\delta) < t < t_0 + \tau(\delta), \quad (\nu = 0, \pm 1, \pm 2, \cdots). \tag{25}$$

According to (25) and (21) it is clearly * impossible that the limit (12) should be a constant function of ξ in the vicinity of the point $\xi = \alpha$.

It is not possible to obtain further results for the asymptotic repartition function of the most general real-valued almost-periodic function. However, in the important special case where the frequencies of the almost-periodic functions are linearly-independent, the method of momentums applied above leads to more refined and explicit results. Cf. a paper to appear in the *Mathematische Zeitschrift*.

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* Analogous considerations have often been used by Bohr; cf., for instance, *Acta Mathematica*, Vol. 45 (1925), p. 55.

ON A GENERALIZATION OF THE LAGRANGE-GAUSS MODULAR ALGORITHM.

By AUREL WINTNER.

Introduction. In the present note the iteration process

$$(1) \qquad a_{n+1} = 2^{-1}(a_n + b_n), \quad b_{n+1} = (a_n b_n)^{1/2}; \qquad (n = 0, 1, 2, \cdots),$$
$$a_0 = a > 0, \quad b_0 = b > 0$$

of the arithmetical-geometrical mean which is the Landen transformation rule of the elliptic modular functions and has furnished the base of the Gaussian theory of elliptic functions will be generalized as follows: a_{n+1} should be not $2^{-1}(a_n + b_n)$ but

$$a_{n+1} = (2^{-1}(a_n{}^\rho + b_n{}^\rho))^{1/\rho}$$

where ρ is any positive number which may be even irrational.

First it will be shown that both limits

$$\lim_{n=\infty} a_n, \qquad \lim_{n=\infty} b_n$$

exist and have a common value which will be denoted by $\mathbf{M}_\rho(a, b)$ inasmuch as it is a function of a, b, and ρ. The value of $\mathbf{M}_1(a, b)$ which belongs to (1) is known to be

$$(1') \qquad \mathbf{M}_1(a, b) = 2^{-1}\pi\,[\int_0^{\pi/2} (a^2 \cos^2\phi + b^2 \sin^2\phi)^{-1/2}\, d\phi]^{-1}.$$

The more general limit $\mathbf{M}_\rho(a, b)$ for which I have not found an analogous explicit representation is also a "mean value" contained in the interval (a, b). The limit $\mathbf{M}_\rho(a, b)$ is then studied as a function of the exponent ρ. Finally the asymptotic behavior of $\mathbf{M}_\rho(a, b)$ for $\lim (b/a) = 0$ is established. There is obtained a direct generalization of the fundamental result of Gauss according to which

$$(1'') \qquad \lim_{b/a=0} \mathbf{M}_1(a, b) \log (4a/b) = a\pi/2$$

and which may be derived from the hypergeometric differential equation of Legendre:

$$\alpha = 1/2, \qquad \beta = 1/2, \qquad \gamma = 1$$

defining the periods of the complete elliptic integral (1'). It is, however, not certain that there exists a hypergeometric differential equation (with more general indices α, β, γ) for which the generalized process ($\rho > 0$) could yield

an analogous explicit Grenzkreisuniformisierung.—The considerations necessary in the case $\rho > 0$ are in some points more complicated than those furnishing the classical theory ($\rho = 1$).

The existence of the generalized Gaussian mean. If A and σ are positive numbers then we understand by A^σ the real branch of the power A^σ so that $A^\sigma > 0$.—First we show that if

$$(2)\quad a_{n+1} = (2^{-1}(a_n{}^\rho + b_n{}^\rho))^{1/\rho}, \quad b_{n+1} = (a_n b_n)^{1/2}; \qquad (n = 0, 1, 2, \cdots)$$

where $\rho > 0$ and

$$(2')\qquad a_0 = a > 0, \qquad b_0 = b > 0$$

then a_n and b_n approach a common limit

$$(3)\qquad \mathbf{M}_\rho(a, b) = \lim_{n=\infty} a_n = \lim_{n=\infty} b_n.$$

Since a_n and b_n are for $a = b$ independent of n we may suppose that

$$(4')\qquad a \neq b.$$

We show that $\mathbf{M}_\rho(a, b)$ lies, for all values of a, b, and ρ, between a and b:

$$(4)\qquad \min(a, b) < \mathbf{M}_\rho(a, b) < \max(a, b).$$

Since the algorithm (2) is homogeneous we have

$$(5)\qquad \lambda \mathbf{M}_\rho(a, b) = \mathbf{M}_\rho(\lambda a, \lambda b); \qquad (\lambda > 0).$$

The function $\mathbf{M}_\rho(a, b)$ fulfills, according to (2) and (3), the functional equation

$$(6)\qquad \mathbf{M}_\rho(a, b) = \mathbf{M}_\rho([2^{-1}(a^\rho + b^\rho)]^{1/\rho}, (ab)^{1/2})$$

which is, of course, a characteristic property of the limit (3).

First we deduce from (2) the identities

$$(7)\qquad 2(a^\rho{}_{n+1} + b^\rho{}_{n+1}) = (a_n{}^{\rho/2} + b_n{}^{\rho/2})^2$$

and

$$(8)\qquad 4(a_{n+1}{}^{2\rho} - b_{n+1}{}^{2\rho}) = (a_n{}^\rho - b_n{}^\rho)^2$$

the second of which may be written in the form

$$(9)\quad 4(a^\rho{}_{n+1} - b^\rho{}_{n+1}) : (a_n{}^\rho - b_n{}^\rho) = (a_n{}^{\rho/2} + b_n{}^{\rho/2})(a_n{}^{\rho/2} - b_n{}^{\rho/2}) : (a^\rho{}_{n+1} + b^\rho{}_{n+1}).$$

On comparing (9) with (7) one obtains

$$2(a^\rho{}_{n+1} - b^\rho{}_{n+1}) : (a_n{}^\rho - b_n{}^\rho) = (a_n{}^{\rho/2} - b_n{}^{\rho/2}) : (a_n{}^{\rho/2} + b_n{}^{\rho/2}),$$

so that

$$| 2(a^\rho{}_{n+1} - b^\rho{}_{n+1}) : (a_n{}^\rho - b_n{}^\rho) | < 1,$$

i. e.
$$| a^\rho{}_{n+1} - b^\rho{}_{n+1} | < 2^{-1} | a_n{}^\rho - b_n{}^\rho | ;$$

hence, according to (2′),

(10)
$$| a_n{}^\rho - b_n{}^\rho | < 2^{-n} | a^\rho - b^\rho | .$$

Furthermore there follows from (8), (4′), (2) and (2′) the inequality

$$a_{n+1}{}^{2\rho} - b_{n+1}{}^{2\rho} > 0,$$

i. e., by virtue of $\rho > 0$, simply

(11a)
$$a_{n+1} > b_{n+1}; \qquad (n = 0, 1, 2, \cdots)$$

so that, according to (2),

(11b)
$$a_{n+2} = (2^{-1}(a^\rho{}_{n+1} + b^\rho{}_{n+1}))^{1/\rho} < (2^{-1}(a^\rho{}_{n+1} + a^\rho{}_{n+1}))^{1/\rho} = a_{n+1}$$

and

(11c)
$$b_{n+2} = (a_{n+1}\, b_{n+1})^{1/2} > (b_{n+1}\, b_{n+1})^{1/2} = b_{n+1}.$$

On collecting (11a), (11b) and (11c) we obtain the inequalities

(11)
$$b_n < b_{n+1} < a_{n+1} < a_n; \qquad (n = 1, 2, \cdots)$$

according to which both sequences

$$a_1, a_2, \cdots; \qquad b_1, b_2, \cdots$$

are monotone and bounded so that the limits $\lim a_n$, $\lim b_n$ necessarily exist. The more precise statement (3) is an obvious consequence of (10). Since, according to (2) and (4′),

(11d)
$$\begin{aligned} a_1 &= (2^{-1}(a^\rho + b^\rho))^{1/\rho} < \max(a, b), \\ b_1 &= (ab)^{1/2} > \min(a, b) \end{aligned}$$

we have, according to (11),

(11′)
$$\min(a, b) < a_n < b_n < \max(a, b).$$

From (11), (11′) and (3) there results the inequality (4).

Continuity of the function $\mathbf{M}_\rho$. We now show that $\mathbf{M}_\rho(a, b)$ is a continuous function of the pair (a, b). It follows readily from (5) that it is sufficient to prove the continuity of $\mathbf{M}_\rho(1, b)$ which is a function of the single variable b and may be represented, according to (3), by the everywhere $(b > 0)$ convergent series

(12)
$$b_1 + \sum_{n=1}^{\infty} (b_{n+1} - b_n).$$

The terms of this series are, according to (2), (2′), (11) and (11′), positive continuous functions of b alone ($a = 1$). From our inequalities there obviously follows the uniform convergence of (12) in any finite range

$$0 < b < \text{const.} \tag{12′}$$

On using complex arguments a, b it would be easy to prove that $\mathbf{M}_\rho(a, b)$ is not only continuous but also analytic.

The mean $\mathbf{M}_\rho(a, b)$ *as a function of* ρ. If a and b have fixed values then the numbers (2) are functions of ρ only, say

$$a_n = a_n(\rho), \qquad b_n = b_n(\rho)$$

In order to show that

$$a_n(\rho) < a_n(\sigma) \quad \text{for} \quad \rho < \sigma;\ n > 1 \tag{13}$$

and

$$b_n(\rho) < b_n(\sigma) \quad \text{for} \quad \rho < \sigma;\ n > 1 \tag{14}$$

we notice the well-known inequality

$$(2^{-1}(A^\rho + B^\rho))^{1/\rho} < (2^{-1}(A^\sigma + B^\sigma))^{1/\sigma} \quad \text{for} \quad \rho < \sigma \tag{15}$$

which is valid for any pair of positive numbers A, B, provided $A \neq B$. For

$$A = a, \qquad B = b$$

we obtain from (15)

$$a_1(\rho) < a_1(\sigma) \tag{13′}$$

and from (11d) there follows

$$b_1(\rho) = b_1(\sigma). \tag{14′}$$

Since $a_1(\rho)$ is, according to (11), different from $b_1(\rho)$ we may apply inequality (15) for

$$A = a_1(\rho), \qquad B = b_1(\rho)$$

and obtain

$$a_2(\rho) = (2^{-1}(a_1(\rho)^\rho + b_1(\rho)^\rho)^{1/\rho} < (2^{-1}(a_1(\rho)^\sigma + b_1(\rho)^\sigma)^{1/\sigma},$$

hence, according to (13′) and (14′),

$$a_2(\rho) < (2^{-1}(a_1(\sigma)^\sigma + b_1(\sigma)^\sigma))^{1/\sigma},$$

i. e.

$$a_2(\rho) < a_2(\sigma). \tag{13a}$$

Since, according to (2),

$$b_2(\rho) = (a_1(\rho)b_1(\rho))^{1/2}, \qquad b_2(\sigma) = (a_1(\sigma)b_1(\sigma))^{1/2},$$

we have, from (13′) and (14′),

$$b_2(\rho) < b_2(\sigma). \tag{14a}$$

Inequalities (13) and (14) accordingly hold for $n = 2$. On proceeding in this manner we obtain that (13) and (14) are generally valid.

There follows now from (13), (14) and (3) that $\mathbf{M}_\rho(a, b)$ *is a monotone function of* ρ. Since $\mathbf{M}_\rho(a, b)$ is, according to (4), a bounded function of ρ we conclude that *the limits*

$$\mathbf{M}_0(a, b) = \lim_{\rho=0} \mathbf{M}_\rho(a, b), \qquad \mathbf{M}_\infty(a, b) = \lim_{\rho=\infty} \mathbf{M}_\rho(a, b) \tag{16}$$

exist.

The retrograde Legendre chain of the complementary moduli for the means $\mathbf{M}_\rho$. We now suppose [cf. (4′)] that

$$a > b \ (> 0) \tag{17}$$

and put

$$c = (a^{2\rho} - b^{2\rho})^{1/2\rho} = c_0 \tag{18}$$

and in a more general manner

$$c_n = (a_n{}^{2\rho} - b_n{}^{2\rho})^{1/2\rho}; \qquad (n = 0, 1, 2, \cdots). \tag{19}$$

We have then from (2′), (17), (11) and (11′)

$$0 < b_n < b_{n+1} < a_{n+1} < a_n; \qquad (n = 0, 1, 2, \cdots) \tag{20}$$

and from (19)

$$c_n > 0; \qquad (n = 0, 1, 2, \cdots). \tag{21}$$

From (19) and (8) there follows

$$4c_{n+1}{}^{2\rho} = 4(a_{n+1}{}^{2\rho} - b_{n+1}{}^{2\rho}) = (a_n{}^\rho - b_n{}^\rho)^2 \tag{22}$$

i. e.

$$c_{n+1} = (2^{-1}(a_n{}^\rho - b_n{}^\rho))^{1/\rho}. \tag{23}$$

On substituting the values (2) in the following formulae [which result from (2) and (23)]:

$$a_{n+2} = (2^{-1}(a^\rho{}_{n+1} + b^\rho{}_{n+1}))^{1/\rho}, \quad c_{n+2} = (2^{-1}(a^\rho{}_{n+1} - b^\rho{}_{n+1}))^{1/\rho} \tag{23′}$$

one obtains after an easy reduction the identities

$$a_{n+2}{}^{\rho/2} = 2^{-1}(a_n{}^{\rho/2} + b_n{}^{\rho/2}), \quad c_{n+2}{}^{\rho/2} = 2^{-1}(a_n{}^{\rho/2} - b_n{}^{\rho/2}) \tag{24}$$

according to which

$$a_{n+2}{}^{\rho/2} + c_{n+2}{}^{\rho/2} = a_n{}^{\rho/2}. \tag{25}$$

Furthermore, we have from (19)

$$c_n^{2\rho} = a_n^{2\rho} - b_n^{2\rho} = (a_n^{\rho} + b_n^{\rho})(a_n^{\rho} - b_n^{\rho}),$$

so that, according to (2) and (23),

$$c_n^{2\rho} = 4a^{\rho}_{n+1}\, c^{\rho}_{n+1}. \tag{26}$$

There follows finally from (2) and (23) the relation

$$a^{\rho}_{n+1} + c^{\rho}_{n+1} = a_n^{\rho}. \tag{27}$$

Since, according to (27) and (26),

$$a_{n-1} = 2^{1/\rho}\,[2^{-1}(a_n^{\rho} + c_n^{\rho})]^{1/\rho}, \quad c_{n-1} = 2^{1/\rho}\,[(a_n\, c_n)^{1/2}]$$

the functional equations (5) and (6) yield

$$\mathbf{M}_{\rho}(a_{n-1}, c_{n-1}) = 2^{1/\rho}\,\mathbf{M}_{\rho}(a_n, c_n)$$

so that, according to (2′) and (18),

$$\mathbf{M}_{\rho}(a, c) = 2^{n/\rho}\,\mathbf{M}_{\rho}(a_n, c_n)\,; \qquad (n = 0, 1, 2, \cdots). \tag{28}$$

On the other hand, from (2) and (6) there follows simply

$$\mathbf{M}_{\rho}(a, b) = \mathbf{M}_{\rho}(a_n, b_n)\,; \qquad (n = 0, 1, 2, \cdots). \tag{29}$$

Since all numbers a_n, b_n, c_n are positive, we may put

$$h_n = 2^{-n\rho} \log\,(2^{2/\rho}\, a_n/c_n)\,; \qquad (n = 0, 1, 2, \cdots) \tag{30}$$

so that [cf. (18)]

$$h_0 = \log\,(2^{2/\rho}/[1 - (b/a)^{2\rho}]^{1/2\rho}) \tag{30′}$$

where we take the real branch of the logarithm. Since, from (26),

$$1/c_{n+1} = 2^{2/\rho}\, a_{n+1}/c_n^{\,2} \tag{26′}$$

equation (30) yields

$$h_{n+1} = 2^{-(n+1)\rho} \log\,(2^{2/\rho}\, a_{n+1}/c_{n+1}) = 2^{-(n+1)\rho} \log\,[(2^{2/\rho}\, a_{n+1}/c_n)^2]$$

i. e.

$$2^{\rho-1}\, h_{n+1} = 2^{-n\rho} \log\,(2^{2/\rho}\, a_{n+1}/c_n), \tag{31′}$$

so that, by virtue of (30),

$$2^{\rho-1}\, h_{n+1} - h_n = 2^{-n\rho} \log\,(a_{n+1}/a_n)\,; \qquad (n = 0, 1, 2, \cdots). \tag{31}$$

On solving the recursion formula (31) by complete induction one obtains the explicit representation

$$2^{(\rho-1)n}\, h_n = h_0 + \sum_{m=1}^{n-1} 2^{-m} \log\,(a_{m+1}/a_m) \tag{32}$$

which assures the existence of the limit

$$\Lambda = \lim_{n=\infty} 2^{(\rho-1)n} h_n = h_0 + \sum_{m=1}^{\infty} 2^{-m} \log (a_{m+1}/a_m) \tag{33}$$

inasmuch as

$$0 < \lim_{m=\infty} a_m = \lim_{m=\infty} a_{m+1}, \quad \text{so that} \quad \lim_{m=\infty} \log (a_{m+1}/a_m) = 0.$$

On solving the recursion formulae (24) and (25) by complete induction and taking the limit for $n = \infty$ Gauss has obtained, in his case $\rho = 1$, the power development of the elliptic theta functions.

The asymptotic behavior of $\mathbf{M}_\rho(1, \kappa)$ *for* $\lim \kappa = 0$. We may now write equation (28), by means of the homogeneity property (5), in the form

$$2^{-n} = [\mathbf{M}_\rho(1, c_n/a_n) : \mathbf{M}_\rho(a/a_n, c/a_n]^\rho \tag{34}$$

and substitute the identities (30) and (34) in the limiting relation (33) obtaining

$$\Lambda = \lim_{n=\infty} \{[\mathbf{M}_\rho(1, c_n/a_n) : \mathbf{M}_\rho(a/a_n, c/a_n)]^\rho \log (2^{2/\rho} a_n/c_n)\}. \tag{35}$$

Since we have, by virtue of (3) and the continuity of the function $\mathbf{M}_\rho$,

$$\lim_{n=\infty} \mathbf{M}_\rho(a/a_n, c/a_n) = \mathbf{M}_\rho(a/\mathbf{M}_\rho(a, b), c/\mathbf{M}_\rho(a, b))$$

i. e. [cf. (5)]

$$\lim_{n=\infty} \mathbf{M}_\rho(a/a_n, c/a_n) = \mathbf{M}_\rho(a, c) : \mathbf{M}_\rho(a, b) \tag{36}$$

so that the relation (35) is equivalent with

$$[\mathbf{M}_\rho(a, b) : \mathbf{M}_\rho(a, c)]^{-\rho} \Lambda = \lim_{n=\infty} \{[\mathbf{M}_\rho(1, c_n/a_n)]^\rho \log (2^{2/\rho} a_n/c_n)\} \tag{37}$$

where, according to (3), (4), and (19),

$$\lim_{n=\infty} c_n = 0, \qquad \lim_{n=\infty} a_n = \mathbf{M}(a, b) > 0.$$

Since the ratio c_n/a_n may take, according to (3), (2) and (19), *any* sufficiently small positive value we may write

$$p_\rho = \lim_{\epsilon=0} \{\mathbf{M}_\rho(1, \epsilon)[\log (2^{2/\rho} \epsilon^{-1})]^\rho\}$$

where ϵ is a *continuous* independent variable. *This relation determines the order of vanishing of the function* $\mathbf{M}_\rho(1, \kappa)$ *of the independent variable* κ *for* $\lim \kappa = +0$. The numerical value of the number p_ρ is known only for $\rho = 1$, viz.

$$p_1 = \pi/2$$

[cf. (1″)].

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ON THE DISTRIBUTIONS OF CERTAIN STATISTICS.*

By ALLEN T. CRAIG.

Introduction. It is the purpose of this paper to present certain general theories on the distributions of such statistics as the arithmetic mean, harmonic mean, geometric mean, median, quartile, decile and range of samples of n items selected at random from a rather arbitrary universe. We shall also develop, without appeal to n-dimensional geometry the distribution of the Pearson χ^2. Exact distributions are known only for a few statistics and these in case the sampled universes are of special types. For a normal parent population, the distribution of the means of samples of n was probably known to Gauss, while Czuber † seems first to have obtained the distribution of the sum of the squares of the deviations of the n items of the sample from the sample mean. Czuber's derivation was closely related to that of Helmert ‡ who dealt with the distribution of sums of squares of deviations from a fixed point. These results of Helmert and Czuber were apparently unknown to Student § who found the distributions of the standard deviation and of the ratio of the deviation of the mean of the sample from the population mean to the standard deviation of the sample in case the parent population is normal. Student's results, somewhat empirically obtained, were later verified by Karl Pearson.¶ By geometrical methods, R. A. Fisher also verified Student's distribution and obtained the distributions of the correlation coefficient,‖ the correlation ratio,** the regression coefficient,** the partial correlation coefficient †† and the multiple correlation coefficient ‡‡

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* Presented to the Society, December 31, 1930.

† E. Czuber, *Theorie der Beobachtungsfehler* (1891), pp. 161-162.

‡ Helmert, "Über die Wahrscheinlichkeit der Potenzsummen," *Zeitschrift für Mathematik und Physik*, Bd. 21 (1876).

§ Student, "The Probable Error of a Mean," *Biometrika*, Vol. 6 (1908), pp. 1-25.

¶ K. Pearson, "Standard Deviations of Small Samples," *Biometrika*, Vol. 10 (1914-15), pp. 522-529.

‖ R. A. Fisher, "Frequency Distribution of the Values of the Correlation Coefficient, etc.," *Biometrika*, Vol. 10 (1914-15), pp. 507-521.

** R. A. Fisher, "The Goodness of Fit of Regression Formulae and the Distribution of Regression Coefficients," *Journal of the Royal Statistical Society*, Vol. 85 (1922).

†† R. A. Fisher, "The Distribution of the Partial Correlation Coefficient," *Metron*, Vol. 3 (1924), pp. 329-333.

‡‡ R. A. Fisher, "The General Sampling Distribution of the Multiple Correlation Coefficient," *Proceedings of the Royal Society*, Vol. 121 (1928).

for normal universes. Much of Fisher's work has been given analytic treatment by Romanovsky.* With the exception of the arithmetic mean, very little is known regarding the exact distributions of statistics obtained from samples taken from non-normal universes. Church † has shown that arithmetic means of samples of n items drawn from a universe represented by a Pearson Type III curve are distributed in accordance with a curve of the same type. This result was later found by Irwin ‡ and C. C. Craig.§ In the same paper, Irwin gave the distribution of arithmetic means of samples of n from a rectangular universe, a result obtained simultaneously by Hall.¶ Both Irwin's and Hall's work on the rectangular universe was considerably antedated by Laplace ‖ and Rietz,** who gave the distribution of the sums of n items from this type of universe. Irwin †† later extended his method of integral equations to obtain the distributions of means of samples of n items drawn from universes represented by Types I and VII of the Pearson system of frequency curves. Baker ‡‡ has treated the distributions of the arithmetic mean and standard deviation for certain non-homogeneous populations and in a later paper §§ extended the method to a universe which can be represented by a finite number of terms of a Gram-Charlier series.

1. *The probability function.* In the present paper, we shall understand a probability function $f(x)$ of a real variable x to be, for all values of x on a range X, a single-valued, non-negative, continuous function with

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* V. Romanovsky, "On the Moments of Standard Deviations, etc.," *Metron*, Vol. 5 (1925), Part 4, pp. 3-45; "On the Criteria That Two Given Samples Belong to the Same Normal Population," *Metron*, Vol. 7 (1928), Part 3, pp. 3-46; "On the Distribution of the Regression Coefficient, etc.," *Bulletin de l'Académie des Sciences de l'U. R. S. S.* (1926).

† A. E. R. Church, "On the Means and Squared Standard Deviations, etc.," *Biometrika*, Vol. 18 (1926), pp. 321-394.

‡ J. O. Irwin, "On the Frequency Distributions of Means, etc.," *Biometrika*, Vol. 19 (1927), pp. 225-239.

§ C. C. Craig, "Sampling When the Parent Population is of Pearson's Type III," *Biometrika*, Vol. 21 (1929), pp. 287-289.

¶ P. Hall, "The Distribution of Means for Samples of Size N, etc.," *Biometrika*, Vol. 19 (1927), pp. 240-245.

‖ Laplace, *Théorie Analytique des Probabilités*, Troisieme Ed., (1820), pp. 257-263.

** H. L. Rietz, "On a Certain Law of Probability of Laplace," *Proceedings of the International Mathematical Congress at Toronto* (1924), pp. 795-799.

†† J. O. Irwin, "On the Frequency Distributions of Means, etc.," *Metron*, Vol. 8 (1930), pp. 51-105.

‡‡ G. A. Baker, "Random Sampling from Non-Homogeneous Populations," *Metron*, Vol. 8 (1930), pp. 67-87.

§§ G. A. Baker, "Distribution of the Means, etc.," *Annals of Mathematical Statistics*, Vol. 1 (1930), pp. 199-204.

$\int_X f(x)\,dx = 1$. Then $\int_a^b f(x)\,dx$ is the probability that a value of x selected at random lies in interval (a, b) where a and b are in X and $a < b$; and $f(x)\,dx$ is, to within infinitesimals of order higher than that of dx, the probability that a value of x selected at random lies in the interval $(x, x + dx)$. It will prove convenient to classify probability functions according as X is the range $(-\infty, \infty)$, $(0, \infty)$, or $(0, A)$, $A > 0$. In accord with this classification, and by adopting the language of Bachelier * to some extent, we shall refer to probability functions as of the first, second, and third kinds respectively.

2. *The distribution of the arithmetic mean.*

THEOREM I.† *Let $f(x)$ be the probability function of a variable x. Let $F(y)$ be that of the sum $y = \Sigma t_i$ where t_i $(i = 1, 2, \cdots n)$ are n independent values of x. If $f(x)$ is a probability function of the first kind, $F(y)$ is a probability function of the first kind, and*

$$(1) \qquad F(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(t_1)f(t_2) \cdots f(t_{n-1})f(y - t_1 - \cdots - t_{n-1})\,dt_{n-1}\cdots dt_1.$$

Moreover, the probability function $\phi_1(\bar{x})$ of $\bar{x} = y/n$ is a probability function of the first kind and

$$(1') \qquad \phi_1(\bar{x}) = nF(n\bar{x}).$$

Proof. By the definition of $F(y)$, $\int_y^{y+\Delta y} F(t)\,dt$ is the probability that y lies in the interval $(y, y + \Delta y)$. For y assigned, it is clear that $t_1, t_2, \cdots, t_{n-1}$ may be chosen arbitrarily from X whereas t_n must be selected with certainty from the interval $(y - t_1 - \cdots - t_{n-1},\ y + \Delta y - t_1 - \cdots - t_{n-1})$ in order that the sum may lie in the interval $(y, y + \Delta y)$. The probability of the joint occurrence of these events is then the product of the probabilities of the separate occurrences integrated over the prescribed intervals. Hence,

$$\int_y^{y+\Delta y} F(t)\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{y-t_1-\cdots-t_{n-1}}^{y+\Delta y-t_1-\cdots-t_{n-1}} f(t_1)f(t_2)\cdots f(t_n)\,dt_n\cdots dt_1.$$

Integrate both members of the equation and denote the primitive of a function by writing a bar over it. We have

* L. Bachelier, *Calcul des Probabilités*, p. 155.

† Cf. R. von Mises, "Fundamentalsätze der Wahrscheinlichkeitsregnung," *Mathematische Zeitschrift*, Vol. 4 (1919), pp. 20, 21, 77. Also C. V. L. Charlier, "Contributions to the Mathematical Theory of Statistics," *Arkiv for Matematik, Astronomi och Fysik*, Bd. 8, No. 4 (1912), pp. 7-9.

$$\bar{F}(y+\Delta y)-\bar{F}(y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(t_1)\cdots f(t_{n-1})$$
$$\times[\bar{f}(y+\Delta y-t_1-\cdots-t_{n-1})-\bar{f}(y-t_1-\cdots-t_{n-1})]\,dt_{n-1}\cdots dt_1.$$

Next, divide both members by Δy and pass to the limit as Δy approaches zero. This establishes the main part of the theorem. Since $F(y)dy$ is, to within infinitesimals of higher order, the probability that a value of y chosen at random lies in the interval $(y, y+dy)$, the change of variable $y=n\bar{x}$ yields the probability function $nF(n\bar{x})$ of $\bar{x}$.

If $f(x)$ is a probability function of the second kind, it is clear that for y assigned, we may choose t_1 arbitrarily from the interval $(0, y)$; t_2 from the interval $(0, y-t_1)$; $\cdots$ and t_{n-1} from the interval $(0, y-t_1-\cdots-t_{n-2})$. Finally, t_n must be selected with certainty from the interval

$$(y-t_1-\cdots-t_{n-1},\ y+\Delta y-t_1-\cdots t_{n-1}),$$

in order that the sum may lie in the interval $(y, y+\Delta y)$. Accordingly, for a probability function of the second kind, (1) takes the form

$$(1.1)\qquad F(y)=\int_0^y\int_0^{y-t_1}\cdots\int_0^{y-t_1-t_2\cdots-t_{n-2}}f(t_1)\cdots f(t_{n-1})$$
$$\times f(y-t_1-\cdots-t_{n-1})\,dt_{n-1}\cdots dt_1$$

while (1′) is unaltered.

We find it convenient to treat the case of a probability function of the third kind in more detail. Let us suppose $f(x)$ to be defined for values of x on the range $(0, a)$. We shall first exhibit $F(y)$ for sums of $n=2, 3, 4$ values. The limits of integration may be easily verified. In this manner we avoid a lengthy exposition for sums of n values. We have for $n=2$,*

$$(1.2)\qquad F(y)=\int_0^y f(t_1)f(y-t_1)\,dt_1,\qquad (0\le y\le a),$$
$$=\int_{y-a}^{a}f(t_1)f(y-t_1)\,dt_1,\qquad (a\le y\le 2a);$$

for $n=3$,†

$$(1.3)\qquad F(y)=\int_0^y\int_0^{y-t_1}f(t_1)f(t_2)f(y-t_1-t_2)\,dt_2dt_1,\qquad (0\le y\le a),$$
$$=\left[\int_0^{y-a}\int_{y-a-t_1}^{a}+\int_{y-a}^{a}\int_0^{y-t_1}\right]f(t_1)f(t_2)f(y-t_1-t_2)\,dt_2dt_1,$$
$$(a\le y\le 2a),$$

* Cf. Czuber, *loc. cit.*, p. 69. Also K. Mayr, "Wahrscheinlichkeitsfunktionen und ihre Anwendungen," *Monatshefte für Mathematik und Physik*, Band 30 (1920), p. 20.

† Czuber, *loc. cit.*, p. 73.

$$= \int_{y-2a}^{a} \int_{y-a-t_1}^{a} f(t_1) f(t_2) f(y - t_1 - t_2)\, dt_2 dt_1, \qquad (2a \leq y \leq 3a);$$

for $n = 4$,

$$(1.4) \qquad F(y) = \int_0^y \int_0^{y-t_1} \int_0^{y-t_1-t_2} \Theta(y, t_1, t_2, t_3)\, dt_3 dt_2 dt_1, \qquad (0 \leq y \leq a)$$

where $\Theta(y, t_1, t_2, t_3) = f(t_1) f(t_2) f(t_3) f(y - t_1 - t_2 - t_3)$

$$= \Big[\int_0^{y-a} \int_0^{y-a-t_1} \int_{y-a-t_1-t_2}^{a} + \int_0^{y-a} \int_{y-a-t_1}^{a} \int_0^{y-t_1-t_2} + \int_{y-a}^{a} \int_0^{y-t_1} \int_0^{y-t_1-t_2} \Big] \Theta(y, t_1, t_2, t_3)\, dt_3 dt_2 dt_1, \qquad (a \leq y \leq 2a),$$

$$= \Big[\int_0^{y-2a} \int_{y-2a-t_1}^{a} \int_{y-a-t_1-t_2}^{a} + \int_{y-2a}^{a} \int_0^{y-a-t_1} \int_{y-a-t_1-t_2}^{a} + \int_{y-2a}^{a} \int_{y-a-t_1}^{a} \int_0^{y-t_1-t_2} \Big] \Theta(y, t_1, t_2, t_3)\, dt_3 dt_2 dt_1, \qquad (2a \leq y \leq 3a).$$

The results of extending the forms of expressions of $F(y)$ in terms of integrals, which we have obtained for $n = 2, 3, 4$, to higher values of n now becomes fairly obvious. For example, with sums of k values, $F(y)$ is defined on $(0, a)$ by a $(k-1)$-fold integral; on $(a, 2a)$ by the sum of $\binom{k-1}{1}$ $(k-1)$-fold integrals; and in general on $[ra, (r+1)a]$ by the sum of $\binom{k-1}{r}$ such integrals. With this in mind and by employing the following abbreviated notations

$$\Theta(y, t_1, \cdots, t_{n-1}) = f(t_1) f(t_2) \cdots f(y - t_1 - \cdots - t_{n-1}),$$
$$x_{ij} = y - ia - t_1 - \cdots - t_j,$$

$$\int_0^y \int_0^{y-t_1} \cdots \int_0^{y-t_1-t_2-\cdots-t_{n-2}} \Theta(y, t_1, \cdots, t_{n-1})\, dt_{n-1} \cdots dt_1$$
$$= \begin{pmatrix} x_{00} & x_{01} & x_{02} & \cdots & x_{0,n-2} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \Theta(y, t_1, \cdots, t_{n-1})\, dt_{n-1} \cdots dt_1,$$

we readily see that for sums of n values

$$(1.5) \qquad F(y) = \begin{pmatrix} x_{00} & x_{01} & x_{02} & \cdots & x_{0,n-2} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \Theta(y, t_1, \cdots t_{n-1})\, dt_{n-1} \cdots dt_1, \qquad (0 \leq y \leq a),$$

$$
\begin{aligned}
=\Bigg[&\begin{pmatrix} x_{10} & x_{11} & \cdots & x_{1,n-3} & a \\ 0 & 0 & & 0 & x_{1,n-2} \end{pmatrix} \\
&+\begin{pmatrix} x_{10} & x_{11} & \cdots & x_{1,n-4} & a & x_{0,n-2} \\ 0 & 0 & & 0 & x_{1,n-3} & 0 \end{pmatrix} \\
&+\begin{pmatrix} x_{10} & x_{11} & \cdots & x_{1,n-5} & a & x_{0,n-3} & x_{0,n-2} \\ 0 & 0 & & 0 & x_{1,n-4} & 0 & 0 \end{pmatrix} \\
&+ \\
&\vdots \\
&+\begin{pmatrix} x_{10} & x_{11} & a & x_{03} & \cdots & x_{0,n-2} \\ 0 & 0 & x_{12} & 0 & & 0 \end{pmatrix} \\
&+\begin{pmatrix} x_{10} & a & x_{02} & \cdots & x_{0,n-2} \\ 0 & x_{11} & 0 & & 0 \end{pmatrix} \\
&+\begin{pmatrix} a & x_{01} & \cdots & x_{0,n-2} \\ x_{10} & 0 & & 0 \end{pmatrix}\Bigg] \Theta(y, t_1, \cdots, t_{n-1})\, dt_{n-1} \cdots dt_1, \\
&\qquad\qquad (a \leqq y \leqq 2a), \\
=\Bigg[&\begin{pmatrix} x_{20} & x_{21} & \cdots & x_{2,n-4} & a & a \\ 0 & 0 & & 0 & x_{2,n-3} & x_{1,n-2} \end{pmatrix} \\
&+\begin{pmatrix} x_{20} & x_{21} & \cdots & x_{2,n-5} & a & x_{1,n-3} & a \\ 0 & 0 & & 0 & x_{2,n-4} & 0 & x_{1,n-2} \end{pmatrix} \\
&+\begin{pmatrix} x_{20} & x_{21} & \cdots & x_{2,n-6} & a & x_{1,n-4} & x_{1,n-3} & a \\ 0 & 0 & & 0 & x_{2,n-5} & 0 & 0 & x_{1,n-2} \end{pmatrix} \\
&+ \\
&\vdots \\
&+\begin{pmatrix} x_{20} & x_{21} & a & x_{13} & \cdots & x_{1,n-3} & a \\ 0 & 0 & x_{22} & 0 & & 0 & x_{1,n-2} \end{pmatrix} \\
&+\begin{pmatrix} x_{20} & a & x_{12} & \cdots & x_{1,n-3} & a \\ 0 & x_{21} & 0 & & 0 & x_{1,n-2} \end{pmatrix} \\
&+\begin{pmatrix} a & x_{11} & x_{12} & \cdots & x_{1,n-3} & a \\ x_{20} & 0 & 0 & & 0 & x_{1,n-2} \end{pmatrix} \\
&+\begin{pmatrix} x_{20} & x_{21} & \cdots & x_{2,n-5} & a & a & x_{0,n-2} \\ 0 & 0 & & 0 & x_{2,n-4} & x_{1,n-3} & 0 \end{pmatrix} \\
&+\begin{pmatrix} x_{20} & x_{21} & \cdots & x_{2,n-6} & a & x_{1,n-4} & a & x_{0,n-2} \\ 0 & 0 & & 0 & x_{2,n-5} & 0 & x_{1,n-3} & 0 \end{pmatrix} \\
&+ \\
&\vdots
\end{aligned}
$$

$$+\begin{pmatrix} x_{20} & x_{21} & a & x_{13} & \cdots & x_{1,n-4} & a & x_{0,n-2} \\ 0 & 0 & x_{22} & 0 & & 0 & x_{1,n-3} & 0 \end{pmatrix}$$

$$+\begin{pmatrix} x_{20} & a & x_{12} & \cdots & x_{1,n-4} & a & x_{0,n-2} \\ 0 & x_{21} & 0 & & 0 & x_{1,n-3} & 0 \end{pmatrix}$$

$$+\begin{pmatrix} a & x_{11} & \cdots & x_{1,n-4} & a & x_{0,n-2} \\ x_{20} & 0 & & 0 & x_{1,n-3} & 0 \end{pmatrix}$$

$$+$$

$$\vdots$$

$$+\begin{pmatrix} x_{20} & a & a & x_{03} & \cdots & x_{0,n-2} \\ 0 & x_{21} & x_{12} & 0 & & 0 \end{pmatrix}$$

$$+\begin{pmatrix} a & x_{11} & a & x_{03} & \cdots & x_{0,n-2} \\ x_{20} & 0 & x_{12} & 0 & & 0 \end{pmatrix}$$

$$+\begin{pmatrix} a & a & x_{02} & \cdots & x_{0,n-2} \\ x_{20} & x_{11} & 0 & & 0 \end{pmatrix}\Bigg] \Theta(y, t_1, \cdots, t_{n-1})\, dt_{n-1} \cdots dt_1, \qquad (2a \leq y \leq 3a),$$

$$\vdots$$

$$= \begin{pmatrix} a & a & \cdots & a \\ x_{n-1,0} & x_{n-2,1} & & x_{1,n-2} \end{pmatrix} \Theta(y, t_1, \cdots, t_{n-1})\, dt_{n-1} \cdots dt_1, \qquad [(n-1)\, a \leq y \leq na].$$

As in the preceding cases, the distribution function of the arithmetic mean $\bar{x} = y/n$ is obtained from that of y by means of (1′).

As applications of the foregoing theorems, we shall now find in as simple algebraic form as possible the distribution functions of the sums and arithmetic means of n items drawn from various types of simple non-normal parent populations.

1. Consider $\qquad f(x) = 1/a, \qquad (0 \leq x \leq a).$

This is a very special case of the theory, and as reference has been made in the introduction to the distribution function of sums of n items drawn from this type of universe, we shall not take the space to give the results.

2. Consider $\qquad f(x) = (k/\sigma)e^{-x/\sigma}, \qquad (0 \leq x < \infty).$

By (1.1), $\qquad F(y) = [k^n y^{n-1}/\sigma^n (n-1)\,!]\, e^{-y/\sigma}$

and $\qquad \phi_1(\bar{x}) = [(kn)^n \bar{x}^{n-1}/\sigma^n (n-1)\,!]\, e^{-n\bar{x}/\sigma}, \qquad (0 \leq \bar{x} < \infty).$

3. Consider $\qquad f(x) = kx^{-\frac{1}{2}} e^{-x/2}, \qquad (0 \leq x < \infty),$

which is the distribution of the squares of items distributed normally * about their expected value in the population. The distribution of sums of n items drawn from this population will then be of special interest as it is the distribution of the Pearson χ^2. Again, by (1.1) we have

$$F(y) = \int_0^y \int_0^{y-t_1} \cdots \int_0^{y-t_1-\cdots-t_{n-2}} (k/t_1^{\frac{1}{2}})(k/t_2^{\frac{1}{2}}) \cdots k/(y - t_1 - t_2 - \cdots - t_{n-1})^{\frac{1}{2}} e^{-y/2}\, dt_{n-1} \cdots dt_1$$

$$= \pi k^n e^{-y/2} \int_0^y (dt_1/t_1^{\frac{1}{2}}) \int_0^{y-t_1} (dt_2/t_2^{\frac{1}{2}}) \cdots \int_0^{y-t_1-t_2-\cdots-t_{n-3}} (dt_{n-2}/t_{n-2}^{\frac{1}{2}})$$

$$= 2\pi k^n e^{-y/2} \int_0^y (dt_1/t_1^{\frac{1}{2}}) \int_0^{y-t_1} (dt_2/t_2^{\frac{1}{2}}) \cdots \int_0^{y-t_1-\cdots-t_{n-4}} (y - t_1 - \cdots - t_{n-3})\,(dt_{n-3}/t_{n-3}^{\frac{1}{2}}).$$

When integrating with respect to t_{n-s}, $s = 3, 4, \cdots, n-1$, make the transformation

$$z = [(y - t_1 - \cdots - t_{n-s-1})/t_{n-s}]^{1/(s-2)}.$$

The integral considered becomes

$$\int_1^\infty (y - t_1 - \cdots - t_{n-s})^{(s-1)/2} \psi(z)\, dz$$

where $\int_1^\infty \psi(z)\, dz$ exists. Accordingly, we find

$$F(y) = k_1 y^{(n-2)/2} e^{-y/2}$$

and

$$\phi_1(\bar{x}) = k_2 \bar{x}^{(n-2)/2} e^{-n\bar{x}/2}, \qquad (0 \leq \bar{x} < \infty).$$

4. Consider

$$f(x) = (k/\sigma) e^{-|x|/\sigma} \qquad (-\infty < x < \infty).$$

For $n = 2$,

$$\phi_1(\bar{x}) = (2k^2/\sigma^2)(\sigma + 2 \mid \bar{x} \mid) e^{-2|\bar{x}|/\sigma}, \qquad (-\infty < \bar{x} < \infty).$$

For $n = 3$,

$$\phi_1(\bar{x}) = (9k^3/2\sigma^3)(\sigma^2 + 3\sigma \mid \bar{x} \mid + 3\bar{x}^2) e^{-3|\bar{x}|/\sigma}, \qquad (-\infty < \bar{x} < \infty).$$

For $n = 4$,

$$\phi_1(\bar{x}) = (2k^4/3\sigma^4)(15\sigma^3 + 60\sigma^2 \mid \bar{x} \mid + 98\sigma\bar{x}^2 + 64 \mid \bar{x}^3 \mid) e^{-4|\bar{x}|/\sigma}, \qquad (-\infty < \bar{x} < \infty).$$

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* H. L. Rietz, "Frequency Distributions Obtained by Certain Transformations of Normally Distributed Variables," *Annals of Mathematics*, Second Series, Vol. 23 (1921-22), p. 294.

5. Consider

$$f(x) = 4x/a^2, \qquad (0 \leq x \leq a/2),$$
$$= (4/a^2)(a - x), \qquad (a/2 \leq x \leq a).$$

For $n = 2$,

$$\phi_1(\bar{x}) = (2^8/3!\,a^4)\bar{x}^3, \qquad (0 \leq \bar{x} \leq a/4),$$
$$= (2^4/3!\,a^4)(-48\bar{x}^3 + 48a\bar{x}^2 - 12a^2\bar{x} + a^3), \qquad (a/4 \leq \bar{x} \leq a/2),$$
$$= (2^4/3!\,a^4)(48\bar{x}^3 - 96a\bar{x}^2 + 60a^2\bar{x} - 11a^3), \qquad (a/2 \leq \bar{x} \leq 3a/4),$$
$$= (2^8/3!\,a^4)(-\bar{x}^3 + 3a\bar{x}^2 - 3a^2\bar{x} + a^3), \qquad (3a/4 \leq \bar{x} \leq a).$$

It is interesting to note that $f(x)$ represents the distribution of means of samples of two and that $\phi_1(\bar{x})$ represents the distribution of means of samples of four for a rectangular parent distribution.

6. Consider

$$f(x) = 2x/a^2, \qquad (0 \leq x \leq a).$$

For $n = 2$,

$$\phi_1(\bar{x}) = (2^5/3a^4)\bar{x}^3, \qquad (0 \leq \bar{x} \leq a/2),$$
$$= (2^4/3a^4)(-2\bar{x}^3 + 3a^2\bar{x} - a^3), \qquad (a/2 \leqq \bar{x} \leqq a).$$

For $n = 3$,

$$\phi_1(\bar{x}) = (3^5/5a^6)\bar{x}^5, \qquad (0 \leq \bar{x} \leq a/3),$$
$$= (3/5a^6)(-162\bar{x}^5 + 270a^2\bar{x}^3 - 180a^3\bar{x}^2 + 45a^4\bar{x} - 4a^5), \qquad (a/3 \leq \bar{x} \leq 2a/3),$$
$$= (3^3/5a^6)(9\bar{x}^5 - 30a^2\bar{x}^3 + 20a^3\bar{x}^2 + 5a^4\bar{x} - 4a^5), \qquad (2a/3 \leq \bar{x} \leq a).$$

2. *The distribution of the harmonic mean.* Let $f(x)$ be the probability function of x. It is well known * that the probability function of $x' = 1/x$ is $F(x') = (1/x'^2)f(1/x')$ provided $1/x$ is continuous on the interval on which $f(x)$ is defined. By use of the theorems on probability functions for sums, we may find $h(y')$, the probability function of $y' = \Sigma\, t'_i$ where t'_i are independent values of x'. The change of variable $y = y'/n$ yields $\psi(y)$, the probability function of the reciprocal of the harmonic mean of samples drawn from the universe represented by $f(x)$. The harmonic mean $\bar{\bar{x}} = 1/y$ then has for its probability function

$$\phi_2(\bar{\bar{x}}) = (1/\bar{\bar{x}}^2)\psi(1/\bar{\bar{x}}). \tag{2}$$

* E. L. Dodd, "The Frequency Law of a Function of One Variable," *Bulletin of the American Mathematical Society*, Vol. 31 (1925), p. 28. Also, "The Frequency Law of a Function of Variables With Given Frequency Laws," *Annals of Mathematics*, Second Series, Vol. 27 (1925-26), p. 18; K. Mayr, *loc. cit.*, p. 20.

As an example, let us consider

$$f(x) = 2a/x^2, \qquad (a \leq x \leq 2a).$$

We find

$$F(x') = 2a, \qquad (1/2a \leq x' \leq 1/a).$$

For $n = 2$,

$$\begin{aligned} h(y') &= 4a^2(y' - 1/a), && (1/a \leq y' \leq 3/2a), \\ &= 4a^2(2/a - y'), && (3/2a \leq y' \leq 2/a), \\ \psi(y) &= 16a^2(y - 1/2a), && (1/2a \leq y \leq 3/4a), \\ &= 16a^2(1/a - y), && (3/4a \leq y \leq 1/a), \end{aligned}$$

and

$$\begin{aligned} \phi_2(\bar{x}) &= (16a^2/\bar{x}^2)(1/a - 1/\bar{x}), && (a \leq \bar{x} \leq 4a/3), \\ &= (16a^2/\bar{x}^2)(1/\bar{x} - 1/2a), && (4a/3 \leq \bar{x} \leq 2a). \end{aligned}$$

For $n = 3$,

$$\begin{aligned} \phi_2(\bar{x}) &= (3^2 4!\, a^3/\bar{x}^2)(1/2\bar{x}^2 - 1/a\bar{x} + 1/2a^2), && (a \leq \bar{x} \leq 6a/5), \\ &= (4!\, a^3/\bar{x}^2)(-9/\bar{x}^2 + 27/2a\bar{x} - 39/8a^2), && (6a/5 \leq \bar{x} \leq 3a/2), \\ &= (3^2 4!\, a^3/\bar{x}^2)(1/2\bar{x}^2 - 1/2a\bar{x} + 1/8a^2), && (3a/2 \leq \bar{x} \leq 2a). \end{aligned}$$

Also consider

$$f(x) = 1/a, \qquad (a \leq x \leq 2a).$$

For $n = 2$,

$$\begin{aligned} \phi_2(\bar{x}) &= [2/a^2]\,[(\bar{x}/2)\log(\bar{x}/[2a - \bar{x}]) + a\bar{x}/2(2a - \bar{x}) - a/2], \\ &\qquad (a \leqq \bar{x} \leqq 4a/3), \\ &= [2/a^2]\,[(\bar{x}/2)\log(4a - \bar{x}/\bar{x}) - a\bar{x}/(4a - \bar{x}) + a], \\ &\qquad (4a/3 \leq \bar{x} \leq 2a). \end{aligned}$$

3. *The distribution of the geometric mean.*

THEOREM II. *Let $f(x)$ be the probability function of the variable x and let $F(y)$ be the probability function of the variable product $y = \Pi\, t_i$ where t_i $(i = 1, 2, \cdots, n)$ are n independent values of x. If $f(x)$ is a probability function of the second kind, $F(y)$ is a probability function of the second kind and*

$$(3) \quad F(y) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{f(t_1)f(t_2)\cdots f(t_{n-1})f[y/(t_1\cdots t_{n-1})]}{t_1\cdots t_{n-1}} dt_{n-1}\cdots dt_1.$$

Moreover, the probability function $\phi_3(z)$ of $z = (y)^{1/n}$ is a probability function of the second kind and

$$(3') \qquad \phi_3(z) = nz^{n-1}F(z^n).$$

Proof. For y assigned, we may select $t_1, t_2, \cdots, t_{n-1}$ arbitrarily from

the interval $(0, \infty)$. But t_n must be selected with certainty from the interval $[y/(t_1 \cdots t_{n-1}), (y+\Delta y)/(t_1 \cdots t_{n-1})]$ in order that the product may lie in $(y, y+\Delta y)$. We then have

$$\int_y^{y+\Delta y} F(t)\,dt = \int_0^\infty \cdots \int_0^\infty \int_{y/(t_1 \cdots t_{n-1})}^{(y+\Delta y)/(t_1 \cdots t_{n-1})} f(t_1) \cdots f(t_n)\,dt_n \cdots dt_1$$

or

$$\bar{F}(y+\Delta y) - \bar{F}(y) = \int_0^\infty \cdots \int_0^\infty f(t_1) \cdots f(t_{n-1}) \times \{\bar{f}[(y+\Delta y)/(t_1 \cdots t_{n-1})] - \bar{f}[y/(t_1 \cdots t_{n-1})]\}\,dt_{n-1} \cdots dt_1.$$

If we assume the validity of the expansion

$$\bar{f}\left(\frac{y}{t_1 \cdots t_{n-1}} + \frac{\Delta y}{t_1 \cdots t_{n-1}}\right) = \bar{f}\left(\frac{y}{t_1 \cdots t_{n-1}}\right) + f'\left(\frac{y}{t_1 \cdots t_{n-1}}\right)\frac{\Delta y}{t_1 \cdots t_{n-1}} + \cdots,$$

the theorem is proved.

If $f(x)$ is a probability function of the third kind, we establish in a precisely similar manner that the probability function $F(y)$ of the product of n independent values of the variable x is

$$(3.1) \quad F(y) = \int_{y/a^{n-1}}^a \int_{y/a^{n-2}t_1}^a \cdots \int_{y/(at_1 \cdots t_{n-2})}^a \frac{f(t_1) \cdots f(t_{n-1}) f[y/(t_1 \cdots t_{n-1})]}{t_1 \cdots t_{n-1}}\,dt_{n-1} \cdots dt_1.$$

Clearly, the distribution of geometric means is given by (3′).

Inasmuch as the geometric mean of n numbers is well-defined only when all of the numbers are positive, this completes our theory of the distribution of this average. As an example, consider

$$f(x) = 1/a, \qquad (0 \leq x \leq a).$$

For $n = 2$,

$$\phi_2(z) = (4z/a^2) \log (a/z), \qquad (0 \leq z \leq a).$$

For $n = 3$,

$$\phi_3(z) = (27z^2/2a^3)\,[\log (a/z)]^2, \qquad (0 \leq z \leq a).$$

4. *The distributions of the median,* quartile and decile.* In a paper by Rider,† the distribution of the medians of samples of $2m+1$ items is given in the case of a rectangular parent distribution. We shall extend the theory to a more general distribution.

Suppose a variable x to obey a law of probability given by $f(x)$. For the

* Cf. Dodd, "Functions of Measurements under General Laws of Error," *Skandinavisk Aktuarietidskrift* (1922), p. 150.

† P. R. Rider, "On the Distribution of the Ratio of Mean to Standard Deviation in Small Samples from Non-Normal Universes," *Biometrika*, Vol. 21 (1929), pp. 136-137.

present we shall assume $f(x)$ to be of the third kind. Let a sample consisting of $n = 2m + 1$ (m an integer) values of x with median value ξ be drawn. We propose to determine the law of probability of ξ. The probability that m of the $2m + 1$ items lie in the interval $(0, \xi)$ is $\binom{2m+1}{m}\left[\int_0^\xi f(t)\,dt\right]^m$. The probability that m of the remaining $m + 1$ items lie in the interval (ξ, a) is $\binom{m+1}{m}\left[\int_\xi^a f(t)\,dt\right]^m$. Finally, the probability that the remaining item lies in $(\xi, \xi + d\xi)$ is $f(\xi)\,d\xi$. We have thus established the following

THEOREM III. *Let $f(x)$ be the probability function of the variable x and let $\phi_4(\xi)$ be the probability function of the median ξ of samples of $n = 2m + 1$. If $f(x)$ is a probability function of the third kind, $\phi_4(\xi)$ is a probability function of the third kind and*

$$\phi_4(\xi) = [(2m+1)!/(m!)^2]\left[\int_0^\xi f(t)\,dt\right]^m\left[\int_\xi^a f(t)\,dt\right]^m f(\xi). \tag{4}$$

If $f(x)$ is a probability function of the first or second kind, $\phi_4(\xi)$ remains unchanged except that the limits of integration are $-\infty$, ξ and ξ, ∞ respectively for a function of the first kind and $0, \xi$ and ξ, ∞ respectively for a function of the second kind.

The probability function of the lower quartile η of samples of $n = 4m + 1$ drawn from a universe represented by $f(x)$ where $f(x)$ is of the second kind, is clearly

$$\phi_5(\eta) = [(4m+1)!/m!(3m)!]\left[\int_0^\eta f(t)\,dt\right]^m\left[\int_\eta^\infty f(t)\,dt\right]^{3m} f(\eta). \tag{4.1}$$

Similar forms of $\phi_5(\eta)$ hold for probability functions of the first and third kinds. It is obvious that the distribution of any statistic which is defined as that value of the variate which exceeds, and is exceeded by, specified numbers of items of the sample, may be determined in this manner. For example, we may find the distribution of the upper quartile or of any decile.

Examples. Consider

$$f(x) = (k/\sigma)e^{-|x|/\sigma}, \qquad (-\infty < x < \infty).$$

Then

$$\phi_4(\xi) = [(2m+1)!/(m!)^2]\,(k^{2m+1}/\sigma)\,e^{-(1/\sigma)(m+1)|\xi|}(2 - e^{-(1/\sigma)|\xi|})^m, \qquad (-\infty < \xi < \infty).$$

Consider $f(x) = (2/a^2)x, \qquad (0 \le x \le a).$

Then

$$\phi_4(\xi) = [(2m+1)!/(m!)^2]\,(2/a^{4m+2})\xi^{2m+1}(a^2 - \xi^2)^m, \qquad (0 \le \xi \le a).$$

Consider $f(x) = e^{-x}, \qquad (0 \le x < \infty).$

Then

$$\phi_4(\xi) = [(2m+1)!/(m!)^2]\, e^{-(m+1)\xi}(1-e^{-\xi})^m, \qquad (0 \leq \xi < \infty).$$

Consider * $\qquad f(x) = 1/a, \qquad (0 \leq x \leq a).$

Then

$$\phi_4(\xi) = [(2m+1)!/(m!)^2] \cdot (1/a^{2m+1})\xi^m(a-\xi)^m, \qquad (0 \leq \xi \leq a).$$

It is interesting to compare the variances of these exact distributions of medians with the variances given by the usual formula † $\sigma_m^2 = 1/4Ny^2$ where N is the number in the sample and y is the ordinate of the probability (not frequency) function of the population at the population median. For the population represented by $f(x) = 1/a$, $\sigma_m^2 = a^2/4(2m+3)$. By the usual formula, $\sigma_m^2 = a^2/4(2m+1)$. For the population represented by $f(x) = 2x/a^2$, we have

$$\sigma_m^2 = \frac{a^2}{2} - \frac{a^2\,[(2m+1)!\,]^6\, 2^{4m+4}}{(m!)^4\,[(4m+3)!\,]^2},$$

while by the ordinary formula $\sigma_m^2 = a^2/8(2m+1)$. It is easy to verify that the limit of

$$\frac{[(2m+1)!\,]^6\, 2^{4m+4}}{(m!)^4\,[(4m+3)!\,]^2}$$

is $1/2$ as m becomes infinite.

5. *The distribution of the range.* In their investigation of test criteria, Neyman and Pearson ‡ found, for a rectangular universe, that the ranges of samples of n items are distributed according to a Pearson Type I curve. In this connection, we shall establish the slightly more general

THEOREM IV. *Let $f(x)$ be the probability function of the variable x and let $\phi_6(W)$ be the probability function of the range W of samples of n items. If $f(x)$ is a probability function of the third kind, $\phi_6(W)$ is a probability function of the third kind and*

$$(5) \qquad \phi_6(W) = n(n-1)\int_W^a f(x)f(x-W)\left[\int_{x-W}^{a} f(x)\,dx\right]^{n-2} dx.$$

Proof. Consider a W assigned within an interval $(W, W+\Delta W)$. The largest item of the sample must be at least as large as W. If x is this largest item, then in order that the range W may lie in the interval $(W, W+\Delta W)$, it is necessary that the smallest item lie in $(x-W-\Delta W,\ x-W)$. The remaining $n-2$ items must lie in the interval $(x-W, x)$. If we consider

* Rider, *loc. cit.*, p. 137.

† H. L. Rietz, *Mathematical Statistics* (1927), p. 135.

‡ J. Neyman and E. S. Pearson, "On the Use and Interpretation of Certain Test Criteria," *Biometrika*, Vol. 20 (1928), p. 210.

all possible arrangements and denote by $\int_W^{W+\Delta W} \phi_6(W)_x dW$ the probability, for x assigned to the interval $(x, x+dx)$, that W lies in the interval $(W, W+\Delta W)$, we have

$$\int_W^{W+\Delta W} \phi_6(W)_x dW = n(n-1)\left[\int_{x-W}^{x} f(x)\,dx\right]^{n-2} f(x)\,dx \int_{x-W-\Delta W}^{x-W} f(x)\,dx$$

whence

$$\phi_6(W)_x = n(n-1)\left[\int_{x-W}^{x} f(x)\,dx\right]^{n-2} f(x)f(x-W)\,dx.$$

Now x may take all values from W to a. Hence, the probability function of W is

$$\phi_6(W) = n(n-1)\int_W^a f(x)f(x-W)\left[\int_{x-W}^{x} f(x)\,dx\right]^{n-2} dx.$$

For the probability functions of the first and second kinds, we find

$$\phi_6(W) = n(n-1)\int_{-\infty}^{\infty} f(x)f(x-W)\left[\int_{x-W}^{x} f(x)\,dx\right]^{n-2} dx,$$

and

$$\phi_6(W) = n(n-1)\int_W^{\infty} f(x)f(x-W)\left[\int_{x-W}^{x} f(x)\,dx\right]^{n-2} dx$$

respectively.

If a sample consists of two items, the standard deviation is $s = \frac{1}{2}\,|\,t_1 - t_2\,|$ or $s = \frac{1}{2}W$. We may, accordingly, obtain the probability function $\phi_7(s)$ of the standard deviation of samples of two drawn from an arbitrary universe by setting $n = 2$ in the above forms of $\phi_6(W)$ and by making the transformation $W = 2s$, $dW = 2ds$.

Examples. Consider *

$$f(x) = 1/a, \qquad (0 \leq x \leq a).$$

We find

$$\phi_6(W) = [n(n-1)/a^{n-1}]W^{n-2}(1 - W/a), \qquad (0 \leq W \leq a),$$

and

$$\phi_7(s) = (4/a^2)(a - 2s), \qquad (0 \leq s \leq a/2).$$

Consider

$$f(x) = e^{-x}, \qquad (0 \leq x < \infty).$$

We find

$$\phi_6(W) = (n-1)e^{(1-n)W}(e^W - 1)^{n-2}, \qquad (0 \leq W < \infty),$$

and

$$\phi_7(s) = 2e^{-2s}, \qquad (0 \leq s < \infty).$$

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* Neyman and Pearson, *loc. cit.*, p. 210; Rider, *loc. cit.*, p. 141.

A CERTAIN TRANSFORMATION ON METRIC SPACES.

By G. T. WHYBURN.

1. In a study of compact locally connected continua Mazurkiewicz † has defined what amounts to a transformation on compact metric spaces K which is effected by changing all distances $\rho(x, y)$ in K into distances $\rho^*(x, y)$ where $\rho^*(x, y)$ is the greatest lower bound of the aggregate of numbers $[\delta(C)]$, where C is any continuum in K containing $x + y$ and $\delta(C)$ is the diameter of C. Now if we alter this definition merely by allowing C to be *any connected set* in K containing $x + y$, it is clear that the transformation will not be altered as applied to compact spaces K. However, in this form it is applicable to spaces which are not compact; and in particular, when applied to metric spaces M which are connected and locally connected, it turns out that, even though in this case the minimum diameter $\rho^*(x, y)$ is not necessarily effectively attained in the sense that there actually exist a connected set C such that $\delta(C) = \rho^*(x, y)$, nevertheless many important properties, including most of those found by Mazurkiewicz and Urysohn in the compact cases, are readily deducible. The present paper is devoted to the development of this transformation to such spaces M and to the application of the result-found to the interesting case of a plane region.

It will be shown, among other results, that if we call this transformation T, then T is a homeomorphism whose inverse is uniformly continuous and which leaves diameters of connected sets invariant. Thus Property S is also invariant under T. Furthermore $T(M)$ is uniformly locally connected, and thus it follows that *any connected, locally connected metric space M is homeomorphic with a space M which is uniformly locally connected.*

In § 5 the results are applied to show that if R is any simply connected plane region having property S ‡ and with boundary B, then the boundary J of $T(R)$ in the space obtained by completing $T(R)$ after the manner of

† See *Fundamenta Mathematicae*, Vol. 1 (1920), pp. 167, 168. This same transformation was also studied later by Urysohn (see *Verhandelingen der Koninklyke Akademie van Wetenschappen te Amsterdam*, Erste Sectie, Deel XIII, No. 4, 1928, pp. 38-42) who also considered it only when applied to compact spaces.

‡ A set of points H has property S provided that for every $\epsilon > 0$, H is the sum of a finite number of connected sets each of diameter $< \epsilon$. (See Sierpinski, *Fundamenta Mathematicae*, Vol. 1, p. 44). It has been shown by R. L. Moore (*Fundamenta Mathematicae*, Vol. 3, p. 235) that a bounded plane simply connected region has property S if and only if its boundary is locally connected.

the Cantor addition of the irrational numbers to the rational number system, is a simple closed curve; and since the inverse of T is uniformly continuous, it can be extended in one and only one way to give a continuous transformation W of J into B. Furthermore, it will be shown that W is a *minimal mapping*, from the standpoint of multiplicity, of J onto B; *the multiplicity under W of any point p of B is equal to the number of components of $B - p$ if this number is finite and is infinite if this number is infinite.* Thus, under the transformation W, not only does the image of a point p on J traverse continuously the points of the boundary B in cyclic order, but the structure of B is accurately described by the transformation W.†

2. *The Transformation T.* We shall consider a connected and locally connected metric space which we denote by M. For each pair of points x and y of M let $\rho^*(x, y)$ be the greatest lower bound of the aggregate of diameters $[\delta(C)]$ of all connected sets C in M which contain both x and y. Now for each pair x, y in M, the number $\rho^*(x, y)$ is a finite non-negative real number. Obviously this is true if $\delta(M)$ is finite, which we may suppose without loss of generality. Even without assuming $\delta(M)$ finite, however, as Mr. C. H. Harry has remarked, we may still show that every $\rho^*(x, y)$ is finite by using the fact that x and y can be joined in M by a finite simple chain of connected regions each of diameter < 1 which is thus a connected set $C \supset x + y$ and with $\delta(C) \leq k$, where k is the number of links in the chain. Also, as will be shown below, the numbers $\rho^*(x, y)$ have distance character in that they satisfy the distance axioms necessary to define a metric space. Let M^* denote the space whose points are exactly the points of M but in which "distance" is defined by means of the above definition, i. e., the distance $\rho(x^*, y^*)$ between two points x^* and y^* of M^* is the number $\rho^*(x, y)$ determined in the above manner for the points x and y of M, where x and y are the points in M identified with the points x^* and y^* respectively in M^*. Thus M^* is the space obtained from M by changing all distances $\rho(x, y)$ into distances $\rho^*(x, y)$. If we regard this operation as a transformation T, then we have that $T(M) = M^*$. We denote points of M by $x, y, \cdots$ and corresponding points in M^* by $x^*, y^*, \cdots$. Distances in each space are denoted by the operator ρ. Thus we have always

$$\text{(i)} \qquad \rho^*(x, y) = \rho(x^*, y^*).$$

We proceed now to establish the following

† In this connection the reader is referred to a result relating the Menger-Urysohn order of a point of a regular curve and its multiplicity under a certain transformation of a circle into that curve, announced by G. Nöbeling in the *Wiener Akademie Anzeiger*, July, 1929, No. 17.

3. *Properties of the transformation.*†

(1) $\rho^*(x, y) \geq \rho(x, y)$;

(2) *diameters of connected sets are invariant under* T, i. e., $\delta(C) = \delta(C^*)$, *where* C *is any connected set;*

(3) *the numbers* $\rho^*(x, y)$ *have distance character,* i. e., $\rho^*(x, y) > 0$, $x \neq y$; $\rho^*(x, x) = 0$; $\rho^*(x, y) = \rho^*(y, x)$; $\rho^*(x, z) \leq \rho^*(x, y) + \rho^*(y, z)$; *thus* M^* *is a metric space;*

(4) T *is a topological transformation. Thus* M^* *is homeomorphic with* M;

(5) *property* S *is invariant under* T;

(6) M^* *is uniformly locally connected;*

(7) *the inverse of* T *is uniformly continuous;*

(8) *if* M *is uniformly locally connected, then* T *is uniformly continuous and conversely;*

(9) *all spherical neighborhoods in* M^* *are connected,* i. e., *for each point* p^* *of* M^* *and each* $r > 0$, *the set* $V_r(p^*)$ *of all points* x^* *in* M^* *with* $\rho(x^*, p^*) < r$ *is connected.*

Proof. If $C \supset x + y$, then $\delta(C) \geq \rho(x, y)$. Therefore, g. l. b. $[\delta(C)] = \rho^*(x, y) \geq \rho(x, y)$, where "g. l. b." means "greatest lower bound," and (1) is proved.

By (1) we have $\delta(C) \leq \delta(C^*)$. Since $x + y \subset C$ implies $\rho^*(x, y) \leq \delta(C)$, therefore we have $\delta(C^*) =$ l. u. b. $[\rho^*(x, y)] \leq \delta(C)$, where x and y are in C and "l. u. b." means "least upper bound." These two inequalities give $\delta(C) = \delta(C^*)$, which is (2).

The first two relations in (3) are corollaries to (1) and (2) respectively. The third results from the fact that $x + y \subset C$ is the same relation as $y + x \subset C$. The fourth follows from the fact that if, in general, C_{ab} denotes a connected set containing $a + b$, then any set $C_{xy} + C_{yz}$ is a set C_{xz}. For then since $\delta[C_{xy} + C_{yz}] \leq \delta(C_{xy}) + \delta(C_{yz})$, we have

$$\text{g. l. b. } [\delta(C_{xz})] \leq \text{g. l. b. } [\delta(C_{xy})] + \text{g. l. b. } [\delta(C_{yz})],$$

which is the same as $\rho^*(x, z) \leq \rho^*(x, y) + \rho^*(y, z)$.

That the transformation T is $(1 - 1)$ follows from (3). By virtue of the local connectivity of M it follows that when $\rho(x_n, x) \to 0$, also $\rho^*(x_n, x) = \rho(x^*_n, x^*) \to 0$; and by (1) we have that $\rho(x_n, x) \to 0$ when $\rho^*(x_n, x) \to 0$.

† For the case of compact spaces, properties (1) and (3) were established by Mazurkiewicz (*loc. cit.*), and property (4) was established by Urysohn (*loc. cit.*).

Therefore T and T^{-1} are continuous and thus T is topological, which proves (4).

If M has property S, then for any $\epsilon > 0$, $M = \sum_1^n C_i$, where each C_i is connected and $\delta(C_i) < \epsilon$. Then since by (2), $\delta(C^*_i) = \delta(C_i)$, we have $M^* = \sum_1^n C^*_i$, where $\delta(C^*_i) < \epsilon$. Thus M^* has property S, and (5) is proved.

To prove (6), let ϵ be any positive number. Take a number $\delta = \epsilon/2$. Then if x^* and y^* are any two points of M^* with $\rho(x^*, y^*) = \rho^*(x, y) < \delta$, by the definition of $\rho^*(x, y)$ there exists a connected set C in M containing $x + y$ and with $\delta(C) < 2\delta = \epsilon$. Then $C^* \supset x^* + y^*$, and, by (2), $\delta(C^*) = \delta(C) < \epsilon$. Thus M^* is uniformly locally connected.

Property (7) follows immediately from (1). For if ϵ is any positive number, then taking $\delta = \epsilon$ we have by (1) that if x^* and y^* are points in M^* with $\rho(x^*, y^*) = \rho^*(x, y) < \delta = \epsilon$, then $\rho(x, y) \leq \rho^*(x, y) < \epsilon$.

If M is uniformly locally connected, then for any $\epsilon > 0$, a $\delta > 0$ exists such that when $\rho(x, y) < \delta$, there exists a connected set C in M with $x + y \subset C$ and $\delta(C) < \epsilon$; thus it follows that $\rho(x, y) < \delta$ implies $\rho^*(x, y) = \rho(x^*, y^*) < \epsilon$, which proves that T is uniformly continuous. If, conversely, T is uniformly continuous, then for any $\epsilon > 0$, a $\delta > 0$ exists such that $\rho(x, y) < \delta$ implies $\rho(x^*, y^*) = \rho^*(x, y) < \epsilon$; therefore, by definition of $\rho^*(x, y)$, there exists a connected set C in M with $x + y \subset C$ and $\delta(C) < \epsilon$, which proves M uniformly locally connected. Thus (8) is established.

Finally, (9) results from the fact that $\rho(x^*, p^*) < r$ implies the existence of a connected set C in M with $x + p \subset C$ and $\delta(C) < r$; this gives $C \subset V_r(p^*)$, from which relation the connectivity of $V_r(p^*)$ is manifest.

4. *Extension of the domain of definition of T.* It has been shown by Hausdorff † that any metric space S may be "completed" into a complete space (i. e., a metric space in which every fundamental sequence of points has a limit point) by adding points on to S corresponding to every fundamental sequence in S which has no limit point in S in a manner entirely parallel to the Cantor method of defining the irrational numbers. Suppose we "complete" our spaces M and M^* in this way and call the resulting complete spaces M_c and M^*_c respectively. We shall consider here three cases.

(a). *In general* since every point of $M^*_c - M^*$ is a limit point of M^* and since, by property (7), T^{-1} is uniformly continuous, it follows by a well known theorem that the definition of T^{-1} can be extended to include the

† See *Grundzüge der Mengenlehre*, 1914, p. 315.

points of $M^*_c - M^*$ in one and only one way so that if we call the resulting transformation W, then W will be single valued and continuous and will be equal to T^{-1} on M^*. The image under W of any point p of $M^*_c - M$ must be a point of $M_c - M$; for if $W(p) = x$ where $x \subset M$, then $x^* \subset M^*$ and hence $x^* \neq p$; but then M^* contains a subset N^* having p but not x as a limit point; hence, W being continuous, the subset $N = W(N^*)$ of M has x as a limit point, contrary to the fact that M and M^* are homeomorphic. Thus we have proved that

(ii) $$W(M^*_c - M^*) \subset M_c - M.$$

(b) *In case M has property S*, then by property (6), M^* also has property S. Then since in the spaces M_c and M^*_c respectively, $M = M_c$ and $\bar{M}^* = M^*_c$ it follows † that M_c and M^*_c have property S. Thus M_c and M^*_c are complete spaces having property S, and hence by a theorem of Hausdorff's,‡ they must be compact. Now in this case it follows that every point of $M_c - M$ must be the image under W of some point of $M^*_c - M^*$. For let p be any point of $M_c - M$. Then since, as is at once clear, $M + p$ is locally connected at p, there exists in M a sequence of connected sets $U_1 \supset U_2 \supset U_3 \supset \cdots$ such that $\delta(U_i) \to 0$ and for each n, $p \subset \bar{U}_n$. For each i, choose a point x_i in U_i. Then since, for each m, $\sum_m^\infty x_i \subset U_i$, it follows that $x^*_1, x^*_2, \cdots$ is a fundamental sequence in M^*. Hence this sequence has a limit point q in M^*_c. The point q cannot be in M^*, for the sequence $x_1, x_2, \cdots$ has as its limit point the point p of $M_c - M$. Thus $q \subset M^*_c - M^*$, and as W is continuous, $W(q) = p$. Thus we have shown that $W(M^*_c - M^*) \supset M_c - M$. By virtue of (ii) we have, in case M has property S, that

(iii) $$W(M^*_c - M^*) = M_c - M.$$

Therefore W is a single valued and continuous transformation mapping M^*

† See R. L. Moore, *loc. cit.*

‡ According to Hausdorff, a space is totally bounded (total beschränkt) provided it is the sum of a finite number of arbitrarily small sets. Hausdorff's theorem which we apply here states that any complete, totally bounded space is compact. See *Grundzüge der Mengenlehre*, 1914, p. 314.

In this connection we note the fact that since clearly the definition of T as well as properties (1), (2), (3), (5), (6), (7) have meaning and are valid whether M is locally connected or not (so long as we suppose, for convenience, that $\delta(M)$ is finite), it follows that *either one of the following conditions is both necessary and sufficient in order that a connected metric space M have property S: (α) that the space M^* be totally bounded or (β) that the space M^*_c be compact.* In case M is compact, property S is equivalent to local connectivity, $M^* = M^*_c$, and this proposition reduces to a theorem of Urysohn (*loc. cit.*).

onto M and $M^*_c - M^*$ onto $M_c - M$. Incidentally we have proved the following theorem:

If M is any connected set having property S, then M is homeomorphic with a uniformly locally connected subset M^* of a compact locally connected continuum M^*_c.

(c). *If M is uniformly locally connected,* then by virtue of (7) and (8) T and T^{-1} are uniformly continuous. Consequently, T and T^{-1} can be extended to the points of $M_c - M$ and $M^*_c - M^*$, respectively, in a unique manner so that the resulting transformations are single valued and continuous on M_c and M^*_c respectively and so that their values on M and M^* are unaltered. Also it is seen at once that under the extended transformations T and T^{-1} it is true that for each point x in M_c (whether x be in M or not), $T^{-1}[T(x)] = x$, so that T^{-1} is the true inverse of T over the completed spaces M_c and M^*_c. Therefore, the extended transformation T is a topological transformation throwing M_c into M^*_c, M into M^*, and $M_c - M$ into $M^*_c - M^*$. Thus we have shown that M_c and M^*_c are homeomorphic and also that $M_c - M$ and $M^*_c - M^*$ are homeomorphic.

5. *Application to plane regions.* Let R be a plane connected region which has property S and which has a connected boundary B. Then R is a space M and, in the notation of this paper, we have at once that $R_c - R = B$. Let J denote the set of points $R^*_c - R^*$, where $R^* = T(R)$. Then, applying § 4, (a) and (b), we have a continuous transformation W of R^*_c into R_c such that $W = T^{-1}$ on R^* and $W(J) = B$. We now proceed to show that

(I) *The set J is a simple closed curve.*

First, J is compact and closed. For by § 4, (b), R^*_c is compact; and if J were not closed, some point x^* of R^* would be a limit point of J. But then as W is continuous, the point x of R would be a limit point of $W(J)$, contrary to the fact that $W(J) = B$ and no point of R is a limit point of B.

Second, J is connected. For if not, then with the aid of the Borel Theorem it is seen that there exists a closed (in R^*_c) and compact subset F^* of R^* such that $R^* - F^* = A^* + G^*$, where A^* and G^* are mutually separated and $\bar{A}^* \cdot J \neq 0 \neq \bar{B}^* \cdot J$ (the bar indicates closure in R^*_c). Then $F \subset R$, $R - F = A + G$, and A and G are mutually separated and $\bar{A} \cdot B \neq 0 \neq \bar{G} \cdot B$. A simple application of the Borel Theorem and of the arcwise connectivity property of R shows the existence of a continuum H in R which contains F in its interior, i. e., every point of F is an interior point of H. Let K be the boundary of the complementary domain D of H

which contains B. By a well known theorem, K is connected. Now since B contains limit points of each of the sets A and G, it follows that D contains points x and y of A and G respectively. But then if xu and yv are arcs in R with $xu \cdot H = u$ and $yv \cdot H = v$, the set $xu + K + yv$ is a connected subset of $R - F$ containing both x and y, contrary to the fact that F separates x and y in R. Therefore J is connected.

Finally, J is disconnected by the omission of any two of its points. For let a and b be any two points of J. Since by (5), R^* has property S, therefore † a and b are accessible from R^*. Accordingly it follows that there exists in R^*_c an arc ab such that $(ab)^* = ab - (a + b) \subset R^*$. Then $W(ab)$ is either an arc or a simple closed curve according as $W(a)$ and $W(b)$ are different or the same and $(ab) = W(ab) - W(a) - W(b) \subset R$. In either case, $R - (ab) = D_1 + D_2$, where D_1 and D_2 are mutually separated non-vacuous sets. Therefore $R^* - (ab)^* = D^*_1 + D^*_2$ and D^*_1 and D^*_2 are mutually separated. Let $J_1 = \bar{D}^*_1 \cdot J - (a + b)$ and $J_2 = \bar{D}^*_2 \cdot J - (a + b)$. Then clearly $J_1 + J_2 = J - (a + b)$. Furthermore, J_1 and J_2 are non-vacuous, mutually separated sets. They are non-vacuous, for there exist points p_1 and p_2 (not necessarily distinct) in $B - W(a) - W(b)$ which are limit points of D_1 and D_2 respectively; D_1 and D_2 contain sequences of points N_1 and N_2 respectively which converge to p_1 and p_2 respectively; since R^*_c is compact, N^*_1 and N^*_2 each have at least one limit point q_1 and q_2 respectively. Then $q_1 + q_2 \subset J$, and since $W(q_1) = p_1$ and $W(q_2) = p_2$, we have $q_1 + q_2 \subset J - (a + b)$; and since q_1 and q_2 are limit points of D^*_1 and D^*_2 respectively, we have $q_1 \subset J_1$ and $q_2 \subset J_2$, which proves J_1 and J_2 non-vacuous. Now if J_1 and J_2 are not mutually separated, it follows that there exists a point p in $J - (a + b)$ which is a limit point both of D^*_1 and of D^*_2. Let U be a neighborhood of p with $U \cdot ab = 0$. Then since, by (b), R^* is uniformly locally connected, it follows that there exist points x_1 and x_2 in D^*_1 and D^*_2 respectively which can be joined by a connected subset of $U \cdot R^*$. But this is impossible because every connected set in R joining D^*_1 and D^*_2 must contain a point of ab. Therefore J_1 and J_2 are mutually separated, and hence J is disconnected by the omission of any two of its points.

Thus we have shown that J is a compact continuum which is disconnected by the omission of any two of its points. It follows by a theorem of R. L. Moore ‡ that J is a simple closed curve.

Since $B = W(J)$ and W is continuous, it follows that B must be locally

† See G. T. Whyburn, *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), p. 650 and *Fundamenta Mathematicae*, Vol. 12, p. 272.

‡ See *Transactions of the American Mathematical Society*, Vol. 21 (1920), p. 342.

connected. Hence as a corollary to (I) we have the result of R. L. Moore's † which states that *The boundary of every plane, connected, and simply connected region which has property S is a locally connected continuum.* Also, in case R is uniformly locally connected and bounded, it will have property S and by § 4, (c), it follows that B and J are homeomorphic. Thus in this case B is a simple closed curve, and as a second corollary to (I) we have R. L. Moore's theorem ‡ to the effect that *The boundary of every bounded, plane, connected, and uniformly locally connected region is a simple closed curve.*

Now let p be any point of B. Then since $W^{-1}(p)$ must cut J into at least as many components as there are components of $B - p$, it follows that the multiplicity $m(p)$ of p under W, i. e., the number of points x on J such that $W(x) = p$, is at least as great as the number $\alpha(p)$ of components of $B - p$. We shall now prove

(II) *If either of the numbers $\alpha(p)$ and $m(p)$ is finite, the other is finite and the two are equal.*

To this end, let $p_1, p_2, p_3, \cdots p_k$ be any set of k-points on J such that, for each $i \leq k$, $W(p_i) = p$. There exist arcs $x^*_i p_i$ such that $x^*_i p_i - p_i \subset R^*$ and such that for each j and $r \leq k$, $\rho\,[(x^*_j, p_j), (x^*_r, p_r)] > d$, where d is some fixed positive number independent of j and r. Then $[W(x^*_i, p_i)]$ are arcs $[x_i p]$ such that $x_i p - p \subset R$ and such that no connected subset of R of diameter $< d$ contains points of two of these arcs. Let C be a circle with center p which is of diameter $< d$ and which does not enclose any of the points $[x_i]$. For each i, the arc $x_i p$ contains an arc $z_i p$ such that $z_i \subset C$ and $z_i p - z_i \subset I$, where I is the interior of C. The arcs $z_1 p, z_2 p, \cdots z_k p$ divide I into exactly k regions $D_1, D_2, \cdots, D_k$. It follows at once that for each i, p is a limit point of $D_i \cdot B$ (for no two of the arcs $z_i p$ can be joined in $I \cdot R$) and, since B is locally connected, there exists in B an arc $u_i p$ such $u_i p - p \subset D_i$. Now no two of the sets $u_i p - p$ can lie together in the same component of $B - p$. For if say $u_i p - p$ and $u_j p - p$ lie together in a component N of $B - p$, then N contains an arc $u_i u_j$ and $u_i u_j + u_i p + u_j p$ contains a simple closed curve L which contains segments $v_i p$ and $v_j p$ of $u_i p$ and $u_j p$ respectively. But then since $v_i p - p \subset D_i$, it follows that both the interior and the exterior of L contain points of R, contrary to the fact that R is connected and $L \cdot R = 0$. Therefore no two of the sets $u_i p - p$ lie together in

† See R. L. Moore, *Fundamenta Mathematicae*, Vol. 3, p. 235.

‡ See R. L. Moore, *Proceedings of the National Academy of Sciences*, Vol. 4 (1918), pp. 364-370.

a single component of $B - p$; and since there are k of the sets $u_i p - p$, it follows that there are at least k components of $B - p$. Thus we have shown that if $W^{-1}(p)$ contains at least k points, $B - p$ has at least k components. Hence $\alpha(p) \geq m(p)$, and if either of these numbers is finite the other is finite and the two are equal.

Finally we prove

(III) *W is not constant on any arc of J.*

To this end, let p be any point of B and let a and b be any two points of J (if two such points exist) such that $W(a) = p = W(b)$. As shown above, there exists an arc ab such that $(ab)^* = ab - (a + b) \subset R^*$. Then $W(ab)$ is a simple closed curve with $W(ab) - p = (ab) \subset R$ and $R - (ab) = D_1 + D_2$, where D_1 and D_2 are mutually separated and D_1 lies within $W(ab)$ and D_2 without $W(ab)$. Let x^* and y^* be points on $(ab)^*$. Then since $x \to p$ and $y \to p$ as $x^* \to a$ and $y^* \to b$ respectively, whereas $\lim_{x,y \to p} \rho^*(x, y) = \rho(a, b)$, it follows that both the interior and the exterior of $W(a, b)$ must contain points q_1 and q_2 respectively of B. Since q_1 is not a limit point of D_2 and q_2 is not a limit point of D_1, it follows that if v_1 and v_2 are points on J such that $W(v_1) = q_1$ and $W(v_2) = q_2$, then v_1 must belong to one of the arcs of J from a to b and v_2 to the other one. Hence W is not constant on either of the arcs of J from a to b, for $W(a) = W(b) = p$, whereas $W(v_1) = q_1$ and $W(v_2) = q_2$ and $q_1 \neq p \neq q_2$; and from this it follows that W is not constant on any arc of J, because a and b were any two points on J having the same image point in B.

By way of recapitulation we note that since $R^* + J$ is homeomorphic with the unit circle plus its interior, the proofs given in this section establish the following

THEOREM. *If R is any plane connected region which has property S and has a connected boundary B, and if C denotes the unit circle and I its interior, then there exists a topological transformation W of I into R which can be extended to the circle C in such a way as to give a single valued and continuous transformation of C into B which is not constant on any arc of C and is such that the multiplicity† under this extended transformation of*

† It seems very probable to the author that the correspondence set up by Carathéodory (See *Mathematische Annalen*, Vol. 73, pp. 323-370) between the prime ends of the boundary of any bounded simply connected region and the points on a circle may, in the case we are here considering, have the same multiplicity relations as we have established for our transformation W.

any point p of B is equal to the number of components of $B - p$ when this number is finite and is infinite when this number is infinite.

In conclusion we remark that, in so far as the set B itself is concerned, we may obtain a mapping W of the circle C onto B having all the properties stated in the above theorem *where B is any compact locally connected continuum containing at least one simple closed curve and every maximal cyclic curve of which is a simple closed curve.* For by a theorem of W. L. Ayres,† any such continuum B, whether it lie in the plane or not, is homeomorphic with the boundary B' of a plane bounded region R. By a theorem of R. L. Moore's ‡ this region R has property S. Hence we can apply our theorem and obtain all the properties for B' and thus also for B, since B and B' are homeomorphic.

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† See Ayres, *Fundamenta Mathematicae*, Vol. 14, p. 92.
‡ See Moore, *Fundamenta Mathematicae*, Vol. 3, p. 232.

ON UPPER SEMI-CONTINUOUS DECOMPOSITIONS OF COMPACT CONTINUA.*

By W. A. WILSON.

1. *Introduction.* The complexities which may occur in the structure of continua have naturally led to various methods of analyzing them. One of the most natural is to decompose the given continuum into sub-continua as elements, which may be taken as "points" of a new locally connected continuum. The use of prime parts for this purpose by H. Hahn and R. L. Moore is well known.

It was with the idea of finding a finer decomposition that the writer invented a notion called the *oscillatory set* (See § 9) of a continuum about a point,† a notion which did not come up to expectations in general on account of a lack of uniqueness. However, in the case of a bounded continuum irreducible between two points it was found that the oscillatory set is unique and that, if no indecomposable continua except continua of condensation are present, the continuum can be exhibited as a simple arc having certain oscillatory sets called *complete* as the "points." The problem for the irreducible continuum was finally completely solved by C. Kuratowski,‡ who divided the continuum into elements called *tranches*, which are identical with the complete oscillatory sets of the writer for the case mentioned above and are connected sets of indecomposable continua and continua of condensation in the more general case.

It is the principal business of the present article to study this same problem for another class of continua for which the oscillatory sets can be taken as unique, namely that of compact one-dimensional continua which have finite connectivity. The space used is metric and compact.

2. *Definitions.* A decomposition of a compact continuum M into disjoint sub-continua $\{X\}$ is called *regular* if (a) $M = \Sigma [X]$; (b) the decomposition is upper semi-continuous; and (c) M is locally connected about

* Presented to the Society, September, 1931.

† "On the Oscillation of a Continuum at a Point," *Transactions of the American Mathematical Society*, Vol. 27 (1925), pp. 429-440; "On the Structure of a Continuum, Limited and Irreducible Between Two Points," *American Journal of Mathematics*, Vol. 48 (1926), pp. 147-168.

‡ "Theorie des continus irreductibles entre deux points II," *Fundamenta Mathematicae*, Vol. 10, pp. 225-275.

each X. The sub-continua $\{X\}$ are called *elements* of M, or elements of the decomposition D of M. The decomposition $M = \Sigma [X]$ is upper semi-continuous if, for every $\epsilon > 0$, there is a $\delta > 0$ such that $X' \subset V_\epsilon(X)$ if $X' \cdot V_\delta(X) \neq 0$.* The continuum M is locally connected about X, if for every $\epsilon > 0$, there is a $\delta > 0$ and a sub-continuum C of M for which $V_\delta(X) \subset C \subset V_\epsilon(X)$.

If D and E are two decompositions of a continuum M, E is said to be *finer* than D if *every* element of E is a sub-set of some element of D and *some* element of E is a *proper* sub-set of an element of D. A sequence $\{D_i\}$ of decompositions of a continuum M is called *descending* if each D_i is finer than D_{i-1}.

If a regular decomposition D is such that there is no finer regular decomposition, we say that the decomposition D is *irreducible*. The next few sections are devoted to a chain of theorems showing that a compact continuum which has any regular decomposition has an irreducible one.

3. As a preliminary it should be noted that, if D and E are upper semi-continuous decompositions of the compact continuum M and E is finer than D, then the decomposition E generates an upper semi-continuous decomposition of at least one element X of the decomposition D.

On the other hand, it is not true that, if D and E are regular, the decomposition of the element X generated by E is necessarily regular. To see this, consider the four plane sets defined as follows:

P: $x = 0$, $(0 \leqq y \leqq 1)$;
Q: $x = 1/2^n$, $(n = 1, 2, \cdots)$; $0 \leqq y \leqq 1$;
R: $y = 0$, $1/2$, or 1; $(0 \leqq x \leqq 1)$;
S: $y = m/2^n$, $m < 2^n$ and integral; $(0 \leqq x \leqq 1/2^n$; $n = 2, 3, 4, \cdots)$.

Then $M = P + Q + R + S$ is a compact locally connected continuum. If we take $P + Q + R$ as one element and the remaining points of M as the other elements, we have a regular decomposition D. If we then take the points of M as elements, we have a regular decomposition E of M which is finer than D. The points of the continuum $P + Q + R$ are the elements into which E decomposes it, but $P + Q + R$ is not locally connected.

4. THEOREM. *Let $\{D_i\}$ be a descending sequence of decompositions of the compact continuum M. For any point x of M let X_i be the element containing x in the decomposition D_i and X be the divisor of the sequence*

* The notation $V_\epsilon(X)$ signifies the set of points of M whose distances from X are less than ϵ. Owing to the compactness of M there is no loss in generality in using the same δ for every X.

$\{X_i\}$. *Then* $M = \mathfrak{S}[X]$ *is a decomposition* D *of* M *into disjoint continua and* D *is finer than every* D_i.

Proof. By definition of X_i and X, each point x of M lies in one and only one X_i; hence $M = \mathfrak{S}[X]$ is disjoint.

Each X is the divisor of a monotone decreasing sequence of compact continua $\{X_i\}$. Hence $M = \mathfrak{S}[X]$ is a decomposition D of M into disjoint continua.

By construction D is at least as fine as every D_i. Suppose that $D = D_i$ for some i. Since D_{i+1} is finer than D_i, some $X_{i+1} \subset X_i$ and $X_{i+1} \neq X_i$. But $X \subset X_{i+1}$, and so $X \neq X_i$, a contradiction. Hence D is finer than every D_i.

This completes the proof. For brevity we shall say that D is the limit of the sequence of decompositions $\{D_i\}$.

5. THEOREM. *Let* $\{X_i\}$ *be a monotone decreasing sequence of sub-continua of a compact continuum* M, *let* M *be locally connected about each* X_i, *and* X *be the divisor of the sequence* $\{X_i\}$. *Then* M *is locally connected about* X.

Proof. Let $\epsilon > 0$ and $\eta = \epsilon/3$. Then for some i_0,

$$X_i \subset V_\eta(X), \qquad i > i_0. \tag{1}$$

On the other hand, for a fixed $i > i_0$, there is a $\delta > 0$ and a sub-continuum F of M such that

$$V_\delta(X_i) \subset F \subset V_\eta(X_i). \tag{2}$$

Since $X \subset X_i$, (1) and (2) show that $V_\delta(X) \subset F \subset V_\epsilon(X)$, which gives the theorem.

COROLLARY. *Let* M *be a compact continuum and* x *be a point of* M. *Then there is a sub-set* X *of* M *irreducible with respect to the properties of being a continuum containing* x *and having* M *locally connected about* X.

6. THEOREM. *Let* $\{D_i\}$ *be a descending sequence of upper semi-continuous decompositions of a compact continuum and* D *be its limit. Then* D *is an upper semi-continuous decomposition.*

Proof. Let A be any element of D and A_i be the element of D_i containing A. The sequence $\{A_i\}$ is monotone decreasing and A is its divisor. Take $\epsilon > 0$ and $\eta = \epsilon/3$. For i greater than some i_0,

$$A_i \subset V_\eta(A). \tag{1}$$

Fix i. Since D_i is upper semi-continuous, there is a $\delta > 0$ such that, if X_i is any element of D_i for which $X_i \cdot V_\delta(A_i) \neq 0$,

$$X_i \subset V_\eta(A_i). \tag{2}$$

Now let X be an element of D such that $X \cdot V_\delta(A) \neq 0$ and let X_i be the element of D_i containing X. Then by (1) and (2),

$$X \subset X_i \subset V_\eta(A_i) \subset V_\epsilon(A).$$

Hence D is upper semi-continuous by definition.

7. THEOREM. *Let $\{D_i\}$ be a descending sequence of regular decompositions of a compact continuum and D be its limit. Then D is a regular decomposition.*

This is an immediate consequence of §§ 4, 5, and 6.

8. THEOREM. *Let M be a compact continuum. Then there is an irreducible regular decomposition of M, if there is any regular decomposition.*

Proof. Consider the hyper-space H whose elements are the sub-continua of M with the distance between any two, A and B, defined as the lower bound of the numbers $\{r\}$ such that $A \subset V_r(B)$ and $B \subset V_r(A)$. It is known that, when M is compact, the space H is also compact.

With each regular decomposition D of M, we associate a set H_D in H, whose "points" are the elements $\{X\}$ of D and all the sub-continua of all the elements $\{X\}$. To save words we call the set of "points" $\{X\}$ the frame* of H. Since the decomposition of M is upper semi-continuous, it is easily seen that H_D is closed. Conversely, a closed sub-set H_D of H, which contains a sub-set $\{X\}$ of "points" which in M are disjoint continua forming a regular decomposition of M and whose remaining "points" are in M all the sub-continua of all the continua $\{X\}$ will be called a D-set.

It is easily seen that, if H_D and H_E are D-sets and H_E is a proper part of H_D, then the regular decomposition E is finer than D; and conversely.

Now let $\{H_i\}$ be a decreasing sequence of D-sets and K be the divisor. For each i a point x of M lies on a continuum X_i which is a "point" of the frame of H_i. If X is the divisor of the sequence $\{X_i\}$, the set of "points" $\{X\}$ in H constitutes a frame for K by § 7. Obviously all the sub-continua of the elements $\{X\}$ are "points" of K. Conversely, if Y is a "point" of K, it is one of every H_i and so in M it is a sub-continuum of some one X_i

* Note that this set is not in general a continuum. For the decomposition of M is merely upper semi-continuous and the definition of distance used in defining H is different from that used in treating the space of elements obtained in an upper semi-continuous decomposition.

for each i; hence Y is a sub-continuum of the divisor X of $\{X_i\}$. Thus K is a D-set.

We see now that the property of being a D-set satisfies the conditions of Brouwer's induction theorem and so H contains an irreducible D-set. But then the corresponding regular decomposition of M is also irreducible.

Remarks. The limitations of the above theorem should not be overlooked. Although it has been shown that for a compact continuum there is an irreducible regular decomposition, it has not been shown that there is only one, nor is there any indication as to how the elements are to be found. As far as the writer knows, this problem has been solved only for special cases,—for example, the irreducible continuum. We now turn to another large class of continua for which a solution is at hand.

9. *Oscillatory sets.* In the first article referred to in § 1 an oscillatory set of a compact continuum M about one of its points x was defined as follows. Let $\{\delta_i\}$ be a decreasing sequence of numbers converging to zero. For each i let V_i be the set of points of M whose distances from x are less than δ_i. Let X_1 be a sub-continuum of M irreducible about V_1, X_2 a sub-continuum of X_1 irreducible about V_2, etc. Then the divisor X of the sequence $\{X_i\}$ is an oscillatory set of M about X. Obviously X is a continuum.

In general this is not unique, but this difficulty can be avoided for the class of one-dimensional compact continua of finite connectivity. Such continua we call *m cyclic* *; if $m = 0$, we call them *acyclic*. The principal thing which makes these continua easy to handle, as will appear in the demonstrations, is the fact that the divisor of any two sub-continua of an m cyclic continuum has at most $m + 1$ components and therefore a relative inner point of both sub-continua is a relative inner point of a component of their divisor.

THEOREM. *Let M be an acyclic one-dimensional compact continuum. Then M has a unique oscillatory set about each of its points $\{x\}$.*

Proof. For each $\delta > 0$ there is at least one sub-continuum X_δ of M irreducible about $V_\delta(x)$. Suppose that there were another, Y_δ. Then $X_\delta \cdot Y_\delta$ is not a continuum, because $V_\delta(x) \subset X_\delta \cdot Y_\delta \subset X_\delta$.

But then $X_\delta \cdot Y_\delta$ has more than one component, which is a contradiction by the above remarks. Thus there is a unique sub-continuum of M irreducible about each $V_\delta(x)$, and consequently the oscillatory set X is unique.

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* The number m is one less than P. Alexandroff's connectivitv number. See his paper, "Über kombinatorische Eigenschaften allgemeiner Kurven," *Mathematische Annalen*, Vol. 96, pp. 512-554.

10. It seems probable that, for a compact continuum of finite cyclic number and a given point x, there is a unique sub-continuum irreducible about $V_\delta(x)$, if δ is small enough. This would insure the uniqueness of the oscillatory set. In the absence of such a proof we introduce the following notion. An oscillatory set of a continuum about a point x is called *irreducible* if no proper sub-set is an oscillatory set about x.

THEOREM. *Let M be an m cyclic compact one-dimensional continuum. If C and D are closed sub-sets, each of which contains an oscillatory set about the point x, then $C \cdot D$ has the same property.*

Proof. It is enough to prove this for the case that C and D are oscillatory sets and neither contains the other. Then there are two descending sequences of vicinities $\{V_i\}$ and $\{W_i\}$, each converging to x, and two monotone decreasing sequences of continua $\{C_i\}$ and $\{D_i\}$, irreducible about the respective vicinities $\{V_i\}$ and $\{W_i\}$ and converging to C and D respectively, and for no i does C_i contain D_i or D_i contain C_i.

Let K_1 be the component of $C_1 \cdot D_1$ containing x. Since $C_1 \cdot D_1$ has a finite set of components and contains either V_1 or W_1, there is an i_2 for which $K_1 \supset V_{i_2} + W_{i_2}$. Then $K_1 \cdot C_{i_2} \cdot D_{i_2}$ contains the smaller of the vicinities V_{i_2} and W_{i_2} and hence K_1 contains a continuum E_2 irreducible about the smaller vicinity. Clearly $E_2 \subset C_1 \cdot D_1$.

Let K_2 be the component of $E_2 \cdot C_{i_2} \cdot D_{i_2}$ containing x; as above it contains $V_{i_3} + W_{i_3}$ for some i_3. Let E_3 be the sub-continuum of K_2 irreducible about the smaller of the vicinities V_{i_3} and W_{i_3}. Clearly $E_3 \subset C_{i_2} \cdot D_{i_2}$.

Continuing indefinitely, we define an oscillatory set E, which is the divisor of the sequence $\{E_n\}$ and is contained in $C \cdot D$.

11. THEOREM. *Let M be an m cyclic one-dimensional compact continuum. Then for each point x there is one and only one irreducible oscillatory set.*

Proof. To show the existence of an irreducible oscillatory set it is sufficient by the Brouwer induction theorem to show that, if $\{E_n\}$ is a descending sequence of closed sets of which each contains an oscillatory set about x and E is the divisor of this sequence, then E has the same property.

Let the decreasing sequence $\{\epsilon_i\}$ converge to zero. Then there is some E_n, which we call E_1, a vicinity V_1 of x, and a continuum C_1 chosen from those converging to some oscillatory set about x contained in E_1, such that C_1 is irreducible about V_1 and is contained in $V_{\epsilon_1}(E)$. Likewise there is an E_2 following E_1 in the sequence $\{E_n\}$, a vicinity V_2 of x of radius less than half

that of V_1, and a continuum C_2 chosen from those converging to some oscillatory set about x contained in E_2, such that C_2 is irreducible about V_2 and is contained in $V_{\epsilon_2}(E)$. Obviously this process can be carried on indefinitely.

Let $F_1 = C_1$. If $C_2 \subset C_1$, let $F_2 = C_2$. Clearly C_2 does not contain C_1 as a proper part, because $V_1 \supset V_2$ and C_2 is irreducible about V_2. If C_2 contains points not on C_1, let K_1 be the component of $C_1 \cdot C_2$ which contains x. Then there is a sub-continuum F_2 of K_1 irreducible about some V_i, which we call V_{i_2}. Then $V_{i_2} \subset F_2 \subset V_{\epsilon_2}(E)$.

If $F_2 \supset C_{i_2+1}$, let $F_3 = C_{i_2+1}$. If not, $F_2 \cdot C_{i_2+1}$ has a component K_2 which contains a sub-continuum irreducible about some V_{i_3}. Then $V_{i_3} \subset F_3 \subset V_{\epsilon_3}(E)$. Since every $i_k \geqq k$, a continuation of this gives an oscillatory set F which is the divisor of the monotone decreasing sequence $\{F_k\}$ and, since $\epsilon_k \to 0$, $F \subset E$.

Hence by the Brouwer theorem there is a set irreducible with respect to the property of containing, and therefore of being, an oscillatory set about x. There cannot be two such, as by the previous theorem their divisor contains an oscillatory set.

Definition. A continuum M will be called *regular* if it has a unique irreducible oscillatory set X about each point x and X is contained in every oscillatory set of M about x. The continuum M in the above theorem is clearly regular. Evidently X is the divisor of all the sub-continua of M containing x as a relative inner point.

12. THEOREM. *Let M be a regular compact continuum. Let the sequence $\{x_i\}$ of points of M converge to x, and $\{X_i\}$ and X be the respective irreducible oscillatory sets. Then X contains the upper closed limiting set of the sequence $\{X_i\}$.*

Proof. For any $\epsilon > 0$ there is a $\delta > 0$ and a sub-continuum X_δ for which $X + V_\delta(x) \subset X_\delta \subset V_\epsilon(X)$. For some i_0, x_i lies in $V_\delta(x)$ if $i > i_0$. Hence some $V_\eta(x_i) \subset V_\delta(x)$. Then the previous inclusions show that $X_i \subset V_\epsilon(X)$, which proves the theorem.

13. THEOREM. *Let M be a regular compact continuum, and let M be locally connected about the sub-continuum K. Then K contains the irreducible oscillatory set of M about each point of K.*

Proof. For each $\epsilon > 0$ there is a $\delta > 0$ and a sub-continuum K_δ for which $V_\delta(K) \subset K_\delta \subset V_\epsilon(K)$. If x lies in K, $V_\delta(x) \subset V_\delta(K)$; consequently for each δ the irreducible oscillatory set X about x is contained in K_δ. Hence $X \subset K$.

COROLLARY. *If M satisfies the above hypotheses and K is the irreducible*

oscillatory set about some point x, M is not locally connected about any proper sub-continuum of K containing x.

14. THEOREM. *Let M be a regular compact continuum, and let K be a sub-continuum containing the irreducible oscillatory set of M about each of its points. Then M is locally connected about K.*

Proof. Let x be any point of K and X be its irreducible oscillatory set. Then for any $\epsilon > 0$ there is a $\delta > 0$, depending on x, and a continuum X_δ such that $V_\delta(x) \subset X_\delta \subset V_\epsilon(X)$. By the Borel theorem there is a finite number of the sets $\{V_\delta(x)\}$ whose union covers K; let K_δ be the union of the corresponding continua $\{X_\delta\}$. Then for some $\eta > 0$, $V_\eta(K) \subset K_\delta \subset V_\epsilon(K)$. Thus M is locally connected about K.

15. THEOREM. *Let M be a regular compact continuum, and let $M = \Sigma\,[K]$ be a decomposition into disjoint continua, each of which is the irreducible oscillatory set about some of its points and contains the irreducible oscillatory sets about all of its points. Then the decomposition is a unique irreducible regular decomposition.*

Proof. By § 14, M is locally connected about each K. Let $\{K_i\}$ be a sequence of elements, each containing a point y_i, and let $y_i \to b$, a point of K'. Let x_i be a point about which K_i is the irreducible oscillatory set. For a sub-sequence $\{x_{i_n}\}$ $x_{i_n} \to a$. If A is the irreducible oscillatory set about a, A contains the upper closed limit of the sequence $\{K_{i_n}\}$, by § 12, and consequently $A \supset b$. Since b also lies in K', it follows that $A \subset K'$. As this is true for any convergent subsequence of the points $\{x_i\}$, we see that K' contains the upper closed limit of $\{K_i\}$ and so the decomposition is regular.

Now let $M = \Sigma\,[L]$ be any other regular decomposition whatever. Any element K is an irreducible oscillatory set about some definite point x, which lies in a definite L. Then $K \subset L$ by § 13. Hence the decomposition $M = \Sigma\,[K]$ is a finer decomposition than $M = \Sigma\,[L]$, and is therefore irreducible.

16. THEOREM. *Let M be a regular compact continuum. Then each point x of M lies on a set K_x saturated with respect to the property of being a sub-continuum of M which contains x, is the union of irreducible oscillatory sets, and has each pair of its points on a connected sub-set consisting of a finite or enumerable set of irreducible oscillatory sets.*

Proof. Since the irreducible oscillatory set X about x has the property above described, we have only to show the existence of a saturated set of this nature. As M is compact, every well-ordered increasing sequence of sub-continua is enumerable. Hence we need only to show that, if $\{K_i\}$ is an

increasing sequence of the sets in question, there is another such set containing the union of the sequence.*

Let U be the union of the sequence $\{K_i\}$. Clearly, if U is closed, it has the desired property. If U is not closed, let W be the union of U and all the irreducible oscillatory sets $\{X\}$, for which $X \cdot \bar{U} \neq 0$. Obviously W is connected. We now proceed to show that it is closed. Let $\{x_i\}$ be a sequence of points in W and $x_i \to a$. If an enumerable set of these points lies in U, a lies in W by the definition of W. Let us assume, then, that no x_i lies in U. Then each x_i lies in the irreducible oscillatory set Y_i of some point y_i and $\bar{U} \cdot Y_i \neq 0$. Let b_i be a point of $\bar{U} \cdot Y_i$. For some partial sequence $\{y_i\}$ and $\{b_i\}$ converge to points y and b, respectively. By § 12 the upper closed limit of the sequence $\{Y_i\}$ is contained in Y, the irreducible oscillatory set about y. Hence $a + b \subset Y$. But b lies in $\bar{U}$ and so $Y \subset W$. This shows that a lies in W, or that W is closed.

It remains to show that any two points a and b of W can be joined by an at most enumerable connected chain of irreducible oscillatory sets contained in W. This is obvious, if $a + b \subset U$. Suppose now that a lies in U and b in $W - U$. Then a lies in some K_i and b lies in an irreducible oscillatory set B containing a limiting point of U; that is, there is a sequence $\{a_i\}$ of points of U converging to a point of B. Now a and a_1 lie on a connected sub-set C_1 of U composed of an at most enumerable set of irreducible oscillatory sets, a_1 and a_2 lie on a similar set C_2, etc. If we let C be the union of the sets $\{C_i\}$, C is connected and, since $B \cdot \bar{C} \neq 0$, so is $B + C$. Then $B + C$ is the required connected set. Finally the case that both a and b lie in $W - U$ is readily deduced from this one. In the light of the first paragraph, this completes the proof.

17. The continua defined in the previous section will be called *irreducible elements.* It is evident that: (a) each point x lies on one and only one element K_x; (b) the decomposition of M into irreducible elements is disjoint; (c) each element contains the irreducible oscillatory set of each of its points.

In consequence of § 11, an m cyclic compact one-dimensional continuum M is either itself an irreducible element or can be decomposed uniquely into irreducible elements. In the latter case it may be called *separable.*

18. THEOREM. *Let M be an m cyclic one-dimensional compact continuum and be separable. Then the decomposition of M into irreducible elements is an irreducible regular decomposition and is the only one possible.*

* See F. Hausdorff, *Mengenlehre,* pp. 173, 174.

Proof. Let $M = \Sigma\,[K]$ be the decomposition in question. This is locally connected by § 14. Let $\{K_i\}$ be a sequence of the elements, a_i be a point in K_i, and the sequence $\{a_i\}$ converge to a point a in some element K. If K does not contain the upper closed limit of the sequence $\{K_i\}$, there is a sub-sequence $\{K_{i_n}\}$ of elements, which we call $\{K_n\}$ to simplify the notation, each containing a point b_n, such that the sequence $\{b_n\}$ converges to a point b in some element L differing from K.

Since M is locally connected about each element, there is for any positive ϵ, which we may take less than one-third the distance between K and L, a $\delta > 0$ and continua K_δ and L_δ such that

$$V_\delta(K) \subset K_\delta \subset V_\epsilon(K) \quad \text{and} \quad V_\delta(L) \subset L_\delta \subset V_\epsilon(L).$$

Obviously $K_\delta \cdot L_\delta = 0$. On the other hand there is an n_0 such that, if $n > n_0$, $K_n \cdot K_\delta \neq 0$ and $K_n \cdot L_\delta \neq 0$. Taking $r > m + 1$, we see that $K_\delta + \Sigma\,[K_n]$ and $L_\delta + \Sigma\,[K_n]$, $n = n_0 + 1,\ n_0 + 2,\ \cdots,\ n_0 + r$, are two sub-continua of M whose divisor has more than $m + 1$ components, a contradiction since M is m cyclic. Hence the assumption that K does not contain the upper closed limit of $\{K_i\}$ is false, and the decomposition is upper semi-continuous. It is then regular.

It remains to show that it is finer than any other regular decomposition $M = \Sigma\,[L]$. Let a and b be points of some K lying on different elements of the second decomposition. By the definition of irreducible element, a and b lie on a connected sub-set C of K, which is the union of an at most enumerable set of irreducible oscillatory sets $\{X_i\}$. By § 13 each of these oscillatory sets lies in one and only one L, which we may call L_i. Then the union of the sets $\{L_i\}$ is connected. This is impossible, since they are not all identical and since the sets $\{L\}$ are mutually disjoint.

Hence every K is a sub-set of some L. This proves the theorem.

19. As seen by an example in the second article referred to in § 1, the irreducible regular decomposition is in general effectively finer than that into prime parts. The continuum of irreducible elements obtained is an at most m cyclic continuous curve, and the notions of end-points, cut-points, etc., can be immediately generalized. A more detailed study requires an investigation of the nature of the irreducible elements themselves.

Finally it may be remarked that the conclusions of §§ 17 and 18 are also valid for a continuum irreducible between two points, whether or not one-dimensional or m cyclic. Since Kuratowski's decomposition of such continua into *tranches* was an irreducible regular decomposition, it follows that for such continua the *tranche* and the irreducible element are identical.

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A COMPLETE CHARACTERIZATION OF PROPER PSEUDO *D*-CYCLIC SETS OF POINTS.

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1. A set S of undefined elements, for suggestiveness called "points" is called a *semi-metrical space* provided that for each two elements p, q of S there corresponds a not negative real number, called the "distance" between the points p and q, such that, denoting this number by pq, we have $pq = qp$, while $pq = 0$ if and only if the points p and q are identical. If for two pairs of points p, q; p', q' we have the relation $pq = p'q'$ we shall say that the two pairs of points are *congruent*. A mapping of a set S upon a set S' is called a congruent mapping if to each pair of points of S there corresponds a congruent pair of points of S'. Finally, two sets S and S' are called congruent provided there exists a congruent mapping of one upon the other.

Karl Menger has characterized the n-dimensional euclidean space R_n among general semi-metrical spaces by means of relations between the distances of its points.* He has shown that the R_n has the congruence order $n + 3$. This means that every semi-metrical space, each $n + 3$ points of which is congruent with $n + 3$ points of the R_n, is congruent with a subset of the R_n. It is further shown that each semi-metrical space containing *more than* $n + 3$ points, each $n + 2$ of which is congruent with $n + 2$ points of the R_n, is congruent with a subset of the R_n. The notion embodied in this important result is expressed by saying that the R_n has the *quasi-congruence* order $n + 2$. Thus, if a set of points is such that each $n + 2$ of the points is congruent with $n + 2$ points of the R_n, while the whole set is not congruent with a subset of the R_n, the set must consist of exactly $n + 3$ points. Such sets are called pseudo-euclidean.

Let us denote by S_n the n-dimensional sphere; that is, S_0 is a pair of points, S_1 is a circle (the term "circle" denoting throughout this paper a curve), S_2 is the surface of a sphere is three-space, etc. It has been shown that the space S_n has the congruence order $n + 3$ also; *but it has not the quasi-congruence order* $n + 2$. Thus, for the circle S_1 the sets analogous to

* Menger, "New Foundations of Euclidean Geometry," *American Journal of Mathematics*, Vol. 53 (1931), pp. 721-745. In this paper, Menger has shortened and revised proofs that appear originally in his paper, "Untersuchungen über allegemeine Metrik," *Mathematische Annalen*, Vol. 100, p. 113.

the pseudo-euclidean sets for R_1, which we shall call pseudo d-cyclic sets, are not confined to sets of exactly four points.*

In this paper we plan to characterize completely those *pseudo d-cyclic sets of points that contain neither a convex tripod* nor a *pseudo-linear* quadruple; that is to say, no four of the points contained in the set are such that one point lies between each two of the three others and no non-linear quadruple of the set has all four of its triples linear.† Such pseudo d-cyclic sets will be called *proper.*‡

2. We shall take as our point of departure, the following lemmas, the proofs of which we omit:

LEMMA 1. *In order that three points p_1, p_2, p_3, form a d-cyclic triple (that is, be congruent to three points of a circle of length $2d$) it is necessary and sufficient that no two of the points have a distance greater than d, and that either the points satisfy the relation $p_1p_2 + p_2p_3 + p_3p_1 = 2d$, or one of the points lies between the two others.*

LEMMA 2. *A proper pseudo d-cyclic quadruple does not contain two linear triples.*

This lemma is obtained by showing that a pseudo d-cyclic quadruple can not contain exactly two linear triples, and then showing that a pseudo d-cyclic quadruple contains exactly three linear triples if and only if the four points form a convex tripod. Further, a proper pseudo d-cyclic quadruple can not contain four linear triples; for in this case the quadruple is either linear or pseudo-linear. The quadruple cannot be pseudo-linear since, by hypothesis, it is proper; it cannot be linear, since it would then be d-cyclic and not pseudo d-cyclic. Lemma 2, then, follows.

LEMMA 3. *A d-cyclic quadruple has at least two linear triples.*

From the last two lemmas, we have the theorem:

THEOREM 1. *In order that four points (p_1, p_2, p_3, p_4) be d-cyclic, it is necessary and sufficient that each three of the points be d-cyclic, that the*

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* The distance between two points of S_1 is measured by the length of the shorter arc joining the two points. The length of the circle is taken as $2d$.

† The point q is said to lie between the points p and r provided that $pq + qr = pr$. The three points are said to be *linear*, since an inspection of the Heronian formula for the area of a triangle shows that if one point lies between two others, the three points are congruent to three points of a line. Evidently, a convex tripod can not be imbedded in a circle.

‡ That a pseudo-linear quadruple, with all its triples d-cyclic, may be pseudo d-cyclic was communicated to the writer by Miss Laura Klaufer, of Vienna, who attributed the remark to Mr. Franz Alt.

points do not form a convex tripod, that at least two of the triples contained in the four points be linear; and that in case the four points are pseudo-linear we have $p_1p_3 = p_2p_4 = d$; $p_1p_2 + p_2p_3 = p_1p_3$.

We shall prove first the following lemma.

LEMMA. *A proper pseudo d-cyclic quadruple has none of its triples linear.*

Let the quadruple p_1, p_2, p_3, p_4 be a proper pseudo d-cyclic quadruple. The quadruple can not contain two linear triples according to Lemma 2. In order to show that it does not contain a single linear triple, we suppose that one triple, say p_1, p_2, p_3, is the only linear triple that it contains, and we deduce from this a contradiction.

Since all of the triples are d-cyclic, and the triple p_1, p_2, p_3 is linear, we have the relations

$$(1) \qquad (p_1p_2 + p_2p_3 - p_3p_1)(p_1p_2 - p_2p_3 + p_3p_1)(p_1p_2 - p_2p_3 - p_3p_1) = 0.$$

$$(2) \qquad p_1p_2 + p_2p_4 + p_4p_1 = 2d.$$

$$(3) \qquad p_2p_3 + p_3p_4 + p_4p_2 = 2d.$$

$$(4) \qquad p_1p_3 + p_3p_4 + p_4p_1 = 2d.$$

Now from (2), (3), and (4) we obtain

$$p_1p_2 + p_2p_3 - p_3p_1 = 2d - 2(p_2p_4),$$
$$p_1p_2 - p_2p_3 + p_3p_1 = 2d - 2(p_1p_4),$$
$$p_1p_2 - p_2p_3 - p_3p_1 = -2d + 2(p_3p_4).$$

Substituting in (1) we obtain

$$(d - p_2p_4)(d - p_1p_4)(d - p_3p_4) = 0,$$

whence one of the three distances p_2p_4, p_1p_4, p_3p_4, is equal to d. But if the distance between two points is equal to d, it is evident that any d-cyclic triple containing these two points is linear. Thus, in any one of the three cases above, some triple other than the triple p_1, p_2, p_3 is linear, and the contradiction sought is obtained.

COROLLARY. *If the quadruple* p_1, p_2, p_3, p_4 *does not form a convex tripod, nor a pseudo-linear set, and is such that each triple is d-cyclic, and one of the triples is linear, then another triple is linear, and the quadruple is d-cyclic.*

3. *Characterization of proper pseudo d-cyclic quadruples.* Let us consider the proper pseudo d-cyclic quadruple p_1, p_2, p_3, p_4. Since by the lemma, no one of the triples that the set contains is linear, but all of the triples are d-cyclic, we have the relations:

$$(1) \quad p_1p_2 + p_2p_3 + p_3p_1 = 2d; \qquad (3) \quad p_2p_3 + p_3p_4 + p_4p_2 = 2d;$$
$$(2) \quad p_1p_2 + p_2p_4 + p_4p_1 = 2d; \qquad (4) \quad p_1p_3 + p_3p_4 + p_4p_1 = 2d.$$

Adding the relations (1) and (2) and using (3) and (4) we obtain $p_1p_2 = p_3p_4$. In a similar way we get $p_2p_3 = p_4p_1$ and $p_1p_3 = p_2p_4$. Substitution of these four values in the above four relations reduces each of them to a single relation, $a + b + c = 2d$, where we have written $a = p_1p_2 = p_3p_4$, $b = p_2p_3 = p_4p_1$, $c = p_3p_1 = p_4p_2$.*

Thus, a proper pseudo d-cyclic quadruple is seen to have its opposite distances equal. It is important to observe, however, that no one of the distances a, b, c is the sum of the other two; for in this case, the quadruple would have all of its triples linear, which we have seen is impossible for a proper pseudo d-cyclic quadruple. It is interesting to compare this with the analogous situation in the case of the R_1. It is known that pseudo-linear quadruples (that is, quadruples not congruent to four points of a line, while each of the triples it contains is linear) have their opposite sides equal, and *one of the sides is the sum of the other two.*† Pseudo-linear quadruples, however, may be imbedded in a circle; *but not necessarily in a circle of metrical diameter d.*

We have, then, established the following theorem completely characterizing proper pseudo d-cyclic quadruples:

THEOREM. *A proper pseudo d-cyclic quadruple is characterized by the fact that its opposite distances are equal, that no one of these distances is equal to the sum of the other two, and that the sum of three of the distances, no two of which are opposite, is equal to the length of the circle.*

COROLLARY 1. *There is a double infinity of proper pseudo d-cyclic quadruples.*

COROLLARY 2. *A pseudo d-cyclic quadruple has no two points with a distance equal to d; i. e., no two points are diametral.*

4. *Characterization of proper pseudo d-cyclic quintuples.* The fundamental theorem for these quintuples is proved by the aid of the following two lemmas:

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* These relations might be obtained also by considering the four equations in the six distances determined by the four points. The matrix of the coefficients is seen to be of rank four, and hence there is a double infinity of solutions of the set. The above relations are valid if the hypothesis is weakened to exclude only sets containing convex tripods.

† Menger, "Untersuchungen über allgemeine Metrik," *Mathematische Annalen*, Vol. 100 (1928), p. 127.

LEMMA 1. *A proper pseudo d-cyclic quintuple does not contain any diametrical points.**

Suppose that two of the points, say p_1 and p_5, contained in the proper pseudo d-cyclic quintuple p_1, p_2, p_3, p_4, p_5 are diametral. Then $p_1p_5 = d$, and we have the relations

$$(1) \qquad p_1p_2 + p_2p_5 = d; \qquad p_1p_4 + p_4p_5 = d; \qquad p_1p_3 + p_3p_5 = d.$$

Now, since the circle has the congruence order four, at least one of the quadruples contained in the five points is pseudo d-cyclic. By corollary 2 of section 3, this quadruple does not contain both p_1 and p_5. We may assume the labelling so that the quadruple p_2, p_3, p_4, p_5 is pseudo d-cyclic. Since the set of five points is, by hypothesis, a proper set, the pseudo d-cyclic quadruple p_2, p_3, p_4, p_5 is proper, and we may apply to it the fundamental theorem of section 3. We have

$$p_2p_3 = p_4p_5; \qquad p_2p_5 = p_3p_4; \qquad p_2p_4 = p_3p_5.$$

Writing $p_2p_3 = a$, $p_2p_5 = b$, $p_2p_4 = c$, we have $a + b + c = 2d$.

Substituting in the relations (1) we obtain $p_1p_2 = d - b$; $p_1p_3 = d - c$; $p_1p_4 = d - a$, and all of the ten distances determined by the five points are expressed in terms of a, b, and c.

Consider, now, the quadruple p_1, p_2, p_3, p_4. We observe that the point p_1 *lies between each pair of the other three points*; that is, the four points form a convex tripod. Then the five points p_1, p_2, p_3, p_4, p_5 are seen to be, contrary to the hypothesis, an improper pseudo d-cyclic set. Hence, the assumption that a pair of points contained in the five points is diametral is seen to lead to a contradiction, and the lemma is proved.†

LEMMA 2. *A proper pseudo d-cyclic quintuple does not contain a linear triple.*

To establish this lemma, we assume that some triple contained in the quintuple is linear, and we show that this assumption leads to a contradiction. Suppose, then, that the triple p_1, p_2, p_3 is linear.‡ Then the two quadruples containing this triple, namely, p_1, p_2, p_3, p_4 and p_1, p_2, p_3, p_5, are d-cyclic, by

* This lemma can be proved under the weaker hypothesis that the set contains no convex tripods.

† It is not true, however, that a pseudo d-cyclic quintuple that does not contain any diametral points is necessarily proper. In the latter part of this section, an example is given of an improper pseudo d-cyclic quintuple without diametral points.

‡ It is sufficient, of course, to obtain the contradiction in this case since the same proof will hold, *mutatis mutandis*, if some other triple should be chosen linear.

the corollary in section 2, and hence each contains a second linear triple. We may select p_1, p_2, p_4 in the first quadruple, and p_1, p_2, p_5 in the second quadruple to be linear. Then the quadruple p_1, p_2, p_4, p_5 is evidently d-cyclic.

This leaves two quadruples to be examined; namely, p_1, p_3, p_4, p_5 and p_2, p_3, p_4, p_5. We show now that one of these quadruples is d-cyclic; for, suppose that both quadruples are pseudo d-cyclic. Then, if we denote by p_i, p_j, p_k any one of the triples contained in these two quadruples, we have

$$\text{(a)} \qquad p_ip_j + p_jp_k + p_kp_i = 2d,$$

and using the results of section 3, we have the relations

$$\text{(b)} \qquad \begin{aligned} p_1p_3 &= p_4p_5 = p_2p_3, \\ p_1p_4 &= p_3p_5 = p_2p_4, \\ p_1p_5 &= p_3p_4 = p_2p_5. \end{aligned}$$

But since the three triples p_1, p_2, p_3; p_1, p_2, p_4; p_1, p_2, p_5 are linear, we have

$$\begin{aligned} (p_1p_2 + p_2p_3 - p_3p_1)(p_1p_2 - p_2p_3 + p_3p_1)(p_1p_2 - p_2p_3 - p_3p_1) &= 0, \\ (p_1p_2 + p_2p_4 - p_4p_1)(p_1p_2 - p_2p_4 + p_4p_1)(p_1p_2 - p_2p_4 - p_4p_1) &= 0, \\ (p_1p_2 + p_2p_5 - p_5p_1)(p_1p_2 - p_2p_5 + p_5p_1)(p_1p_2 - p_2p_5 - p_5p_1) &= 0. \end{aligned}$$

Upon substituting from the relations (b) the above three relations yield

$$p_1p_2 = 2(p_1p_3) = 2(p_1p_4) = 2(p_2p_4) = 2(p_2p_5) = 2(p_1p_5).$$

Thus, from (b), the distance p_1p_2 is equal to the double of any one of the other nine distances determined by the five points. Substituting in any one of the relations (a) we see that each of these nine distances is equal to $2d/3$, and hence p_1p_2 is equal to $4d/3$. But this is impossible, since the points p_1 and p_2 are congruent to two points of a circle of length $2d$ and hence their distance can not exceed d.

We have thus shown that one of the quadruples p_1, p_3, p_4, p_5; p_2, p_3, p_4, p_5, is d-cyclic (since the contrary assumption has been seen to lead to a contradiction), and hence contains at least two linear triples. If the triple p_3, p_4, p_5, common to both quadruples, is linear, then the contradiction sought is obtained, for in this case both quadruples would be d-cyclic, and hence all five quadruples contained in the five points would be d-cyclic. But the circle has the congruence order four, and hence if all the quadruples formed from the five points are d-cyclic, the five points are themselves d-cyclic, and not pseudo d-cyclic, as supposed.

Let us select, then, another triple, say p_1, p_4, p_5 to be linear.* Then the

* If either of the remaining triples p_1, p_3, p_5 or p_1, p_2, p_4 is chosen linear, a contradiction due to the presence of diametral points is obtained in a similar manner.

quadruple p_1, p_3, p_4, p_5 is d-cyclic, and hence contains at least one other linear triple. Since we have already investigated the case in which the triple p_3, p_4, p_5 is linear, we suppose that one of the triples p_1, p_3, p_4; p_1, p_3, p_5 is linear. If the first of these triples is linear, then the quadruple p_1, p_2, p_3, p_4 has three of its triples linear, while if the second triple is linear, the quadruple p_1, p_2, p_3, p_5 has three of its triples linear. Now if either of the two remaining fourth triples, namely, p_2, p_3, p_4 and p_2, p_3, p_5, is linear, then the quadruple p_2, p_3, p_4, p_5 is d-cyclic and, as we have seen above, all the quadruples are d-cyclic, and the contradiction is obtained. We suppose, then, that neither of these two triples is linear, and we show that this is also impossible; for if a quadruple is d-cyclic and has three of its triples linear, while its fourth triple is not linear, then the quadruple must have two diametral points.* Hence either the quadruple p_1, p_2, p_3, p_4 or the quadruple p_1, p_2, p_3, p_5 has two diametral points, which is impossible, since by lemma 1 the quintuple, being proper, does not contain two diametral points.†

Thus, the assumption that a proper pseudo d-cyclic quintuple contains a linear triple is seen to lead to a contradiction, and the lemma is proved.

We are now in a position to prove the fundamental theorem for proper pseudo d-cyclic sets containing exactly five points. This theorem we state as follows:

Fundamental Theorem for proper pseudo d-cyclic quintuples. A proper pseudo d-cyclic quintuple is equilateral; that is, each two pairs of its points are congruent.

By lemma 2 we know that the set does not contain any linear triple.‡ Therefore, each of the five quadruples contained in the five points is a proper pseudo d-cyclic quadruple. Applying the theorem of section 3 to each of these five quadruples, we obtain immediately that all of the ten distances determined by the five points are equal, and an inspection of the relation satisfied by each of the ten d-cyclic triples shows that each distance is equal to $2d/3$. Hence, the theorem is proved.

Remark 1. We note, again, that the above theorem was proved for pseudo d-cyclic quintuples not containing a convex tripod nor a pseudo-linear quad-

* To prove this it is necessary to examine twenty-seven cases obtained by considering the possible ways in which the three triples can be linear.

† A similar method yields the desired contradiction in case the quadruple p_2, p_3, p_4, p_5 is chosen as containing a linear triple.

‡ Evidently, this is necessary for the validity of the theorem, for since the line does not contain an equilateral triangle, if one of the triples is linear, the three points forming the triple can not be equilateral.

ruple; that is, for proper pseudo d-cyclic quintuples. The following table furnishes an example of a pseudo d-cyclic quintuple that is *not* equilateral, but the set is not proper, since it contains convex tripods:

	p_1	p_2	p_3	p_4	p_5
p_1		$d/3$	$d/3$	$d/3$	$2d/3$
p_2	$d/3$		$2d/3$	$2d/3$	$d/3$
p_3	$d/3$	$2d/3$		$2d/3$	$d/3$
p_4	$d/3$	$2d/3$	$2d/3$		$d/3$
p_5	$2d/3$	$d/3$	$d/3$	$d/3$	

where the number appearing at the intersection of the i-th row and j-th column is the distance p_ip_j. It is easily verified that each three of the five points is d-cyclic, but the quadruple p_1, p_2, p_3, p_4 is seen to form a convex tripod, and is therefore not d-cyclic. Then, *a fortiori*, the five points are not d-cyclic, and are hence, pseudo d-cyclic. The triple p_2, p_3, p_4 is the only non-linear d-cyclic triple contained in the five points.

Remark 2. A necessary and sufficient condition that a pseudo d-cyclic quintuple contain a linear triple is that the set contain a convex tripod, or a pseudo-linear quadruple.

The sufficiency of the condition is evident, for if the set contains a convex tripod, then three of its triples are linear, and if it contains a pseudo-linear quadruple then four of its triples are linear. The necessity of the condition is seen to follow from the results just obtained, for if a pseudo d-cyclic quintuple contains a linear triple, the quintuple can not be proper (since by the theorem, if it were proper it would be equilateral, and hence would not contain a linear triple). Hence the set is improper; that is, it contains a convex tripod, or a pseudo-linear quadruple.

5. *Proper pseudo d-cyclic sets of n points.* We are now in a position to prove by induction the fundamental theorem characterizing proper pseudo d-cyclic sets containing more than four points. We state the theorem:

THEOREM. *A proper pseudo d-cyclic set containing more than four points is equilateral.*

We have proved the theorem for the case $n = 5$. Let us assume that the theorem is true for $n = k$, $k > 4$. We show that this implies the validity of the theorem for $n = k + 1$.

Consider, then, a proper pseudo d-cyclic set of $k + 1$ points. At least one of the sets of k points contained in these $k + 1$ points is pseudo d-cyclic;

for if every set of k points contained the $k+1$ points is d-cyclic, then, since $k > 4$, every set of four points contained in the $k+1$ points is, *a fortiori*, d-cyclic. But since the circle has the congruence order four, it would follow that the set of $k+1$ points is d-cyclic, and not pseudo d-cyclic as assumed in the hypothesis. We may so label the $k+1$ points that the set

$$p_1, p_2, p_3, \cdots, p_{k-1}, p_k$$

is pseudo d-cyclic. Since the set of $k+1$ points is proper, the above pseudo d-cyclic set is also proper, and hence, by hypothesis, this set is equilateral, with each of the $(1/2)k(k-1)$ distances determined by these k points equal to $2d/3$.

We now show that the set of $k+1$ points must contain at least one other pseudo d-cyclic set of k points. To establish this, we suppose that

$$p_1, p_2, p_3, \cdots, p_{k-1}, p_k$$

is the only pseudo d-cyclic set contained in the $k+1$ points. Then the remaining k sets of k points into which the $k+1$ points of the set can be arranged are all d-cyclic, and hence all of the quadruples contained in these k sets are d-cyclic. Now these k sets may be obtained by omitting, in turn, from the $k+1$ points one of the points of the set $p_1, p_2, \cdots, p_{k-1}, p_k$. The k sets so obtained may be ordered by agreeing to call the i-th set that set in which the point p_i does not appear, $(i = 1, \cdots, k)$. Then all of the quadruples contained in the pseudo d-cyclic set $p_1, p_2, \cdots, p_{k-1}, p_k$ that are independent of p_i are contained in the i-th one of these k sets. Thus every quadruple that can be formed from the k points $p_1, p_2, \cdots, p_{k-1}, p_k$ is to be found in one of the k d-cyclic sets, and hence is itself d-cyclic. But this is impossible, for if every quadruple contained in the set $p_1, p_2, \cdots, p_{k-1}, p_k$ is d-cycle, then the set is d-cyclic, and not pseudo d-cyclic as was supposed. Therefore, at least one other set of k points contained in the $k+1$ points of our set is pseudo d-cyclic.

To fix the ideas, let us suppose that the other pseudo d-cyclic set whose existence we have demonstrated above is the set $p_2, p_3, \cdots, p_k, p_{k+1}$. (If some other set is pseudo d-cyclic, the argument is, of course, entirely similar.) Then this set being proper is, by hypothesis, equilateral, and each of the distances determined by the k points is equal to $2d/3$. But we have already seen that each of the distances determined by the points $p_1, p_2, \cdots, p_{k-1}, p_k$ is equal to $2d/3$. Hence, of the $(1/2)k(k+1)$ distances determined by the $k+1$ points all are seen to be equal to $2d/3$ with the exception of the distance p_1p_{k+1}, which does not enter into these two sets.

To determine this distance, consider any triple containing p_1p_{k+1}, say the triple p_1, p_2, p_{k+1}. It is clear that this triple is not linear, for if it is linear, we have

$$(p_1p_2 + p_2p_{k+1} - p_1p_{k+1})(p_1p_2 - p_2p_{k+1} + p_1p_{k+1}) \times (p_1p_2 - p_2p_{k+1} - p_1p_{k+1}) = 0$$

and upon substituting for p_1p_2 and p_2p_{k+1} their equal, $2d/3$, we obtain $p_1p_{k+1} = 4d/3$, which is impossible. Hence the triple is not linear, and we have

$$p_1p_2 + p_2p_{k+1} + p_1p_{k+1} = 2d,$$

from which it is immediate that $p_1p_{k+1} = 2d/3$. Thus each distance has been shown equal to $2d/3$, and hence the set is equilateral.

COROLLARY 1. *A proper pseudo d-cyclic set of n points does not contain any linear triple.*

COROLLARY 2. *A necessary and sufficient condition that a pseudo d-cyclic set of n points* ($n \geq 4$) *contains a linear triple is that the set be improper.*

COROLLARY 3. *A proper pseudo d-cyclic set of n points does not contain any diametral points.*

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ON SUBSETS OF A CONTINUOUS CURVE WHICH LIE ON AN ARC OF THE CONTINUOUS CURVE.*

By EDWIN W. MILLER

Introduction. In 1919 R. L. Moore and J. R. Kline gave the following *Theorem:* †

If M is a closed and bounded plane point set, in order that M be a subset of an arc (of the plane) it is necessary and sufficient that the following two conditions be satisfied:

1. *Every component of M is an arc or a point.*
2. *No interior point of an arc-component, t, of M is a limit point of $M - t$.*

As the plane is a particular example of a continuous curve,‡ the problem considered by Moore and Kline appears as a special case of the following general problem:

If S is a continuous curve in n-dimensional Euclidian space, E_n, and M is a closed and bounded subset of S, what conditions are necessary and sufficient that M be a subset of an arc of S?

It is with this problem that the present paper is chiefly concerned. The restriction that M be closed is not a vital one, as any set M is a subset of an arc if and only if the same is true of its closure $\bar{M}$.

Part I is given over to a study of the Moore-Kline conditions, which are of exceptional interest because of their purely internal character; that is to say, these conditions may be defined without reference to the imbedding space. As the proof of the above mentioned theorem of Moore and Kline makes use of properties of E_2 which are not properties of $E_n (n > 2)$ a different procedure from that pursued by Moore and Kline is adopted to obtain the result that the Moore-Kline theorem still holds if M is any bounded and closed subset of $E_n (n > 2)$.

In Part II a theorem is given which reduces the general problem for any continuous curve in E_n to the following problem: If a and b are two

* The chief results of Part I of this paper were presented under separate title to the American Mathematical Society on November 30, 1928. Part II was presented to the Society on April 30, 1930.

† *Annals of Mathematics*, Vol. 20 (1919), pp. 218-223.

‡ Throughout this paper the term "continuous curve" is used to mean a locally connected (connected im kleinen) continuum.

points of a cyclicly connected * continuous curve S, and M is a closed subset of S, under what conditions is M a subset of an arc in S whose end points are a and b? This problem remains unsolved as far as the present paper is concerned. However, with the aid of the above mentioned theorem complete solutions of the problem are obtained for certain important classes of continuous curves.

I.

THEOREM 1. *In $E_n (n > 2)$ in order that the closed and bounded set M be a subset of an arc it is necessary and sufficient that*

1. *Each component of M is either an arc or a point.*

2. *If t is an arc-component of M, then no interior point of t is a limit point of $M - t$.*

Proof. The conditions are evidently necessary.

They are also sufficient. L. W. Cohen has shown † that any set M in a separable metric space is homeomorphic with a subset of the linear continuum if and only if four conditions are satisfied, two of which conditions are the conditions given above, and the remaining two of which are:

3. If p is a point-component of M then $dim_p M = 0$; ‡ and

4. If a is an end point of an arc-component t of M then $dim_a (M - t + a) = 0$.

It will be shown that for a closed and bounded set M in E_n (the particular separable metric space here considered) condition (3) is satisfied and that (4) follows from (2). In fact, let F be a closed and bounded set in E_n, K a component of F, ϵ any positive number and H the set of all points of F which are at a distance $\geqq \epsilon$ from K. The set H, as is readily shown, is closed. It follows from a theorem of Miss Anna Mullikin § that F is the sum of two mutually exclusive closed sets F_1 and F_2 such that $F_1 \supset K$ and $F_2 \supset H$. If K is a single point we have the result that a closed set in E_n is 0-dimensional at any one of its point-components. Thus (3) is at once seen to be satisfied. From (2) it follows that if t is any arc-component of M and a and b are the end points of t, then $M - t + a + b$ is a closed set. The point a is evidently a point-component of $M - t + a + b$. Hence $dim_a (M - t$

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* A continuous curve S is *cyclicly connected* if every two points of S lie on some simple closed curve which is a subset of S.

† *Fundamenta Mathematicae*, Vol. 14 (1929), pp. 281-303.

‡ For definitions of dimension, see Menger, *Dimensionstheorie* (Leipzig, 1928).

§ *Transactions of the American Mathematical Society*, Vol. 24 (1922), pp. 144-162.

$+ a + b) = 0$. But $dim_a(M - t + a + b) = dim_a(M - t + a)$. Thus (4) is also satisfied. The set M, then, is homeomorphic with some subset $\bar{M}$ of the linear continuum. Since M is closed and bounded, $\bar{M}$ is a subset of some linear interval I. Let $\bar{a}$ be the first point of $\bar{M}$ on I and $\bar{b}$ the last point of M on I, and consider the interval $(\bar{a}, \bar{b})$. Let us arrange the open intervals complementary to $\bar{M}$ in a sequence $(\bar{a}_i, \bar{b}_i)$. It will be shown that corresponding to every i there is an arc N_i in E_n whose end points, a_i and b_i, are the points of M which correspond to $\bar{a}_i$ and $\bar{b}_i$, and which is such that $W = M + \sum_{i=1}^{\infty} N_i$ is an arc from a to b, the points of M corresponding to $\bar{a}$ and $\bar{b}$.

For this purpose we shall extend the following

Theorem of J. R. Kline: In E_n $(n \geqq 2)$ if H is the sum of a countable infinity of closed sets M_i, such that no $M_i = E_n$, and no M_i disconnects any domain, then if D is a domain, $D - D \cdot H$ is arc-wise connected.*

THEOREM A. *Under the conditions of the above theorem if a and b are any two points of D, there is an arc ab such that $ab - a - b \subset D - D \cdot H$.*

Proof. If $a \in D \cdot H$, then a is a limit point of $D - D \cdot H$. In fact, since the closed set $M_i \neq E_n$, and disconnects no domain of E_n, it is easily shown to be nowhere dense. Hence H is of the first category of Baire, so that a is a limit point of $D - D \cdot H$. We can evidently find a sequence of distinct points $\{a_n\}$ such that $\lim a_n = a$, and such that the sphere

$$S_n = S(a_n, 2d(a_n, a_{n+1}))$$

is a sub-domain of D. By the above theorem of Kline, $S_n - S_n \cdot H$ contains an arc $a_n a_{n+1}$. The point set $a + \sum_{n=1}^{\infty} a_n a_{n+1}$ is evidently a continuous curve containing a_1 and a. There is, then, an arc $a_1 a \subset a + \sum_{n=1}^{\infty} a_n a_{n+1}$ (R. L. Moore)†. Evidently, except for the point a, the arc $a_1 a \subset D - D \cdot H$. From this result Theorem A follows readily.

Now, the set T consisting of all point-components of M together with the end points of all arc-components is clearly a closed, totally disconnected set. Furthermore, as Cohen shows,‡ the arc-components of M are countable.

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* *Bulletin of the American Mathematical Society*, Vol. 23 (1917), pp. 290-292. The conditions here imposed on H are those given by R. L. Wilder, *ibid.*, Vol. 23 (1927), p. 388.

† See, for instance, R. L. Moore, *Bulletin of the American Mathematical Society*, Vol. 23 (1917), pp. 233-236.

‡ *Loc. cit.*

As any closed totally disconnected set is 0-dimensional and any arc is 1-dimensional,* M is the sum of a countable family of closed sets each of dimension $\leqq 1$. The set $M = T + \sum_{i=1}^{\infty} M_i$. Urysohn has shown † that a closed set of dimension $\leqq n-2$ disconnects no domain of E_n. Since we are here concerned with $E_n (n \geqq 3)$, neither T nor any one of the arc-components M_i disconnects any domain of E_n. Evidently neither T nor any one of the sets $M_i = E_n$. Hence the conditions of Theorem A are satisfied. Thus, if $S_1 = S(a_1, 2d(a_1, b_1))$ there is an arc N_1 from a_1 to b_1 such that

$$N_1 - a_1 - b_1 \subset S_1 - S_1 \cdot M.$$

The set $M + N_1$ satisfies the conditions of Theorem A. Thus, if

$$S_2 = S(a_2, 2d(a_2, b_2))$$

there is an arc N_2 from a_2 to b_2 such that

$$N_2 - a_2 - b_2 \subset S_2 - S_2 \cdot (M + N_1).$$

In general, the set $M + \sum_{i=1}^{n-1} N_i$ satisfies the conditions of Theorem A. Hence if $S_n = S(a_n, 2d(a_n, b_n))$ there is an arc N_n from a_n to b_n such that

$$N_n - a_n - b_n \subset S_n - S_n \cdot (M + \sum_{i=1}^{n-1} N_i).$$

It will now be shown that $W = M + \sum_{i=1}^{\infty} N_i$ is an arc.

As $\bar{M}$ is a closed bounded set the mapping of $\bar{M}$ on to M is uniformly continuous. Since the intervals $(\bar{a}_n, \bar{b}_n)$ do not overlap and all lie on a finite interval, $\lim_n d(\bar{a}_n, \bar{b}_n) = 0$. It follows that $\lim_n d(a_n, b_n) = 0$. Therefore $\lim_n \operatorname{diam}(S_n) = 0$ and as $N_n \subset S_n$, $\lim_n \operatorname{diam}(N_n) = 0$. Now let p be a limit point of W. If p is a limit point of M then $p \epsilon W$. If p is a limit point of $\sum_{i=1}^{\infty} N_i$, it either belongs to a particular N_i and therefore to W, or else it is a limit point of a set of points only a finite number of which belong to any given N_i. In this case, since $\lim_n \operatorname{diam}(N_n) = 0$, it is a limit point of the end points of the arcs N_i. But the end points of N_i are points of M. Thus W is closed.

If W is not connected it is the sum of two mutually exclusive non-vacuous closed sets Q_1 and Q_2. Not all the points of M belong to Q_1 (for example

* See Menger, *loc. cit.*

† *Fundamenta Mathematicae*, Vol. 8 (1926), p. 311.

for then we would have $Q_2 = 0$. Let us put, then, $M \cdot Q_1 = M'$ and $M \cdot Q_2 = M''$. The set $M = M' + M''$ where M' and M'' are both closed non-vacuous sets. This separation of M determines a separation of $\bar{M}$ into two mutually exclusive closed non-vacuous sets $\bar{M}'$ and $\bar{M}''$. Let $\bar{p}$ be any point of $\bar{M}'$. As $\bar{p}$ is not a limit point of $\bar{M}''$, and as there is at least one point of $\bar{M}''$ either to the left of $\bar{p}$ or to the right,—say to the right, there will be a first point $\bar{q}$ of the closed set $\bar{M}''$ to the right of $\bar{p}$. Since $\bar{q}$ is not a limit point of $\bar{M}'$ and since there is at least one point of $\bar{M}'$ to the left of $\bar{q}$, viz. $\bar{p}$—there is a last point $\bar{p}'$ of the closed set $\bar{M}'$ to the left of $\bar{q}$. Now $\bar{p}'$ and $\bar{q}$ are the end points of an interval complementary to $\bar{M}$. Therefore there is an arc N_i whose end points are p' and q. Hence $p' + q \subset Q_1$, or else $p' + q \subset Q_2$. It follows that $\bar{p}' + \bar{q} \subset \bar{M}'$ or else $\bar{p}' + \bar{q} \subset \bar{M}''$. This is in contradiction with the way in which $\bar{p}'$ and $\bar{q}$ were chosen.

We now show that $W - p$ is not connected if $W - (a + b) \supset p$.

Case 1. The point p belongs to M.

Let $W - p = W_1 + W_2$, where W_1 consists of all points x of M such that $\bar{x}$ is to the left of $\bar{p}$, together with all points x of W which are interior points of arcs N_i such that $\bar{a}_i$ and $\bar{b}_i$ are not to the right of $\bar{p}$; and where $W_2 = (W - p) - W_1$. The set $W_1 \neq 0$ since $W_1 \supset a$, and $W_2 \neq 0$ since $W_2 \supset b$. Now, let $q_1 \epsilon W_1$. There exists a sphere $S(q_1, d)$ such that $S(q_1, d) \cdot M \cdot W_2 = 0$ since the only limit point of the points of $\bar{M}$ to the right of $\bar{p}$ which is not a point of this set is the point $\bar{p}$. Since $\lim_i \operatorname{diam}(N_i) = 0$, there is a positive integer r such that if $i > r$, then $\operatorname{diam}(N_i) < d/2$. Hence if $i > r$, and if $N_i \subset W_2 + p$, then $S(q_1, d/2) \cdot N_i = 0$. Let F consist of all points z such that $z \epsilon N_i$ where $i \leqq r$. The set F is closed and $F \cdot q_1 = 0$. Since q_1 is neither a limit point of $M \cdot W_2$ nor of $W_2 \cdot \sum_{i=1}^{\infty} N_i$, it is not a limit point of W_2. Similarly no point of W_2 is a limit point of W_1.

Case 2. p is not a point of M.

Then p is a point of some N_i,—say N_k. If $a_k \neq a$, and $b_k \neq b$, then a_k and b_k determine separations: $W - a_k = W_1^{(a_k)} + W_2^{(a_k)}$, and $W - b_k = W_1^{(b_k)} + W_2^{(b_k)}$ in virtue of the result of Case 1. Denote by $a_k p$ the arc of N_k from a_k to p and by $p b_k$ the arc of N_k from p to b_k, and put $W - p = K_1 + K_2$; where $K_1 = (W_1^{(a_k)} + a_k p - p)$ and $K_2 = (p b_k - p + W_2^{(b_k)})$. Since $K_1 \subset W_1^{(b_k)}$ it contains no limit point of $W_2^{(b_k)}$. It evidently contains no limit point of $p b_k - p$. Since $K_2 \subset W_2^{(a_k)}$ it contains no limit point of $W_1^{(a_k)}$. It evidently contains no limit point of $a_k p - p$. Thus $W - p$ is the sum of two mutually separated sets. In case $a_k = a$ or $b_k = b$, we can effect

a separation of $W - p$ in a similar fashion. This concludes the proof that W is an arc.

From Theorem 1 and the Moore-Kline theorem we have at once the following

COROLLARY: *If $M \subset E_n$ and $E_n \subset E_m$, where E_n and E_m are Euclidian spaces of n and m dimensions, then if M is a subset of an arc in E_m it is also a subset of an arc in E_n.*

THEOREM 2. *If M is a closed and bounded subset of a (connected) domain D of E_n, and if M satisfies the conditions of Moore and Kline, then there is an arc A such that $M \subset A \subset D$.*

Proof. By Theorem 1 there is an arc A' in E_n which contains M. If $q \epsilon A' \cdot D$, and C_q is the component of $A' \cdot D$ determined by q, only a finite number of distinct components C_q can contain points of M. For suppose there is an infinity of such components — $C_1, C_2, \cdots, C_n, \cdots$, and let $a_n \epsilon C_n \cdot M$. The points a_n are all distinct, and since M is bounded, the set $[a_n]$ has at least one limit point, p. Since M is closed, $p \epsilon M$ and therefore $p \epsilon D$. The component C_p is evidently a closed, half-closed or open arc, t.*

If t is a closed arc then evidently $A' \subset D$. If t is a half-closed arc denote $t +$ both its end points by t'. That end point a of t' which belongs to t is evidently an end point of the arc A'. In this case, then, p is either an end point of A' or an inner point of t'. But, clearly, then, since A' is an arc, p is not a limit point of $A' - t'$ and therefore not of $[a_n]$. Finally, if t is an open arc, and t' denotes $t +$ its end points, then p is an inner point of t'. But then, p is not a limit point of $A' - t'$, and therefore is not a limit point of $[a_n]$. Hence only a finite number of components C_q of $A' \cdot D$ contain points of M. Since M is a closed subset of D it is apparent that each C_q contains a closed sub-arc containing all the points of $M \cdot C_q$. Thus there exists a finite number of mutually exclusive arcs $A_1, \cdots A_m$ such that $M \subset \sum_{i=1}^{m} A_i \subset D$. It is an easy consequence of the Wilder accessibility theorem † that in E_2 the finite set of arcs $\sum_{i=1}^{m} A_i$ is contained in an arc of D. For E_n with $n > 2$ successive application of Theorem A leads to a like result.

THEOREM 3. *If D is a domain of E_n, and M is a closed bounded subset*

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* An arc is said to be closed if it contains both end points, half-closed if it contains just one end point, and open if it contains neither end point.

† R. L. Wilder, *Fundamenta Mathematicae*, Vol. 7 (1923), pp. 340-377.

of D which satisfies the Moore-Kline conditions, and if p_1 and p_2 are distinct points of M, then there is an arc p_1p_2 such that $M \subset p_1p_2 \subset D$ if and only if.

(1) *p_1 and p_2 are point-components of M; or*

(2) *p_1 and p_2 are end points of distinct arc-components t_1 and t_2 of M such that p_1 is not a limit point of $M - t_1$ and p_2 is not a limit point of $M - t_2$; or*

(3) *one of the points p_1, p_2 is a point-component of M and the other is an end point of an arc-component t of M, and is not a limit point of $M - t$.*

Proof. Since no interior point of an arc-component t of M can be an end point of an arc which contains M, and since the same is true of an end point of an arc-component t of M which is a limit point of $M - t$, the conditions are at once seen to be necessary.

We now show that the conditions are also sufficient. By Theorem 2 there is an arc A such that $M \subset A \subset D$. We may suppose that the end points a_1 and a_2 of A are points of M. Let us first assume that neither a_1 nor a_2 is p_1 or p_2, and consider the case $n = 2$. On the basis of well-known results it is easy to show that there exists a simple closed curve J such that if I is the interior of J, then $J + I \subset D$ and $A \subset J$. If either p_1 or p_2 is an end point of an arc-component t of M and is not a limit point of $M - t$, or else is a point-component of M and is not a limit point of M from both sides on A, the simple closed curve J evidently contains an arc A' which contains M and has one of the two points, p_1 and p_2,—let us say, p_1, as an end point. If both of the points p_1 and p_2 are point-components of M and limit points of M from both sides on A, we shall construct an arc A' which contains M and has p_1 as one of its end points as follows: We may evidently suppose that the order of the points a_1, p_1, p_2, and a_2 is a_1, p_1, p_2, a_2. Let us choose two sequences of sub-arcs of A complementary to M, with end points x_n and y_n and x'_n and y'_n respectively, so that

$$a_1 < x'_{n-1} < y'_{n-1} < x'_n < y'_n < p_1 < x_n < y_n < x_{n-1} < y_{n-1} < p_2,$$

and such that $\lim x'_n = \lim y'_n = \lim x_n = \lim y_n = p_1$.

There is an arc α_1, from x'_1 to x_1 such that $\alpha_1 - x'_1 - x_1 \subset I$. The arc α_1 forms with the sub-arc x'_1x_1 of A a simple closed curve J_1 whose interior $I_1 \subset I$. There is an arc α'_1 from y_2 to y'_1 such that $\alpha'_1 - y_2 - y'_1 \subset I_1$. The arc α'_1 forms with the sub-arc $y_2y'_1$ of A a simple closed curve J'_1 whose interior $I'_1 \subset I_1$.

We proceed in this way to construct arcs α_2 from x'_2 to x_2, α'_2 from y_3 to y'_2, α_3 from x'_3 to x_3, and so on. Now, it is possible to choose the arcs α_n

and α'_n so that $\lim \operatorname{diam}(\alpha_n) = \lim \operatorname{diam}(\alpha'_n) = 0$. In fact, let us prove the following

LEMMA. *If x and y are two points of a simple closed curve J, whose interior is I, and xy is either arc of J from x to y, then there exists an arc α from x to y such that $\alpha - x - y \subset I$, and $\operatorname{diam}(\alpha) < \operatorname{diam}(xy) + \epsilon$, where ϵ is a preassigned positive number.*

Proof. Let D denote the set of points of I at a distance $< \epsilon/2$ from the arc xy. It is easily shown that D is an open and connected subset of I. The points x and y lie on the boundary of D and neither belongs to any continuum of condensation of the boundary. Hence, by the Wilder accessibility theorem, x and y are both accessible from D. It follows at once that there is an arc α in I from x to y such that $\operatorname{diam}(\alpha) < \operatorname{diam}(xy) + \epsilon$.

We will suppose, then, that the arcs α_n and α'_n are chosen so that $\lim \operatorname{diam}(\alpha_n) = \lim \operatorname{diam}(\alpha'_n) = 0$. Then the point set A', consisting of the arc $y_1p_2a_2$ of A, that arc x'_1a_2 of J which does not contain p_1, the arcs α_n and α'_n, and the sub-arcs x_ny_{n+1} and $y'_nx'_{n+1}$ of A $(n = 1, 2, \cdots)$ is, as can easily be shown, an arc from p_1 to y_1 which contains M and lies in D.

An argument which is only a slight modification of the one given above suffices to show that starting with the arc A' we can construct an arc A'' of D which contains M and has p_1 *and* p_2 as its end points. This completes the proof for the case $n = 2$.

For the case $n > 2$, our procedure is similar. By Theorem A, the end points of the arc A are accessible from D. There is then a simple closed curve J such that $A \subset J \subset D$. As in the case $n = 2$, we are reduced to the supposition that p_1 is a point-component of M which is a limit point of M from both sides on A. We choose two sequences of sub-arcs of A complementary to M as before, and construct arcs α_n and α'_n, all mutually exclusive, such that $\lim \operatorname{diam}(\alpha_n) = \lim \operatorname{diam}(\alpha'_n) = 0$. That there exist two such sequences of arcs α_n and α'_n is readily seen to be a consequence of Theorem A. The rest of the proof runs parallel to the proof for the case $n = 2$.

It is natural, in this connection, to consider the problem of determining conditions under which a closed point set M in E_n is a subset of a ray or an open curve.*

THEOREM 4. *A closed set M in E_n is a subset of a ray if and only if the two following conditions are satisfied:*

1. *Either* (*a*) *the components of M are all arcs and points, or* (*b*) *the*

* For definitions of these, see R. L. Moore, *Transactions of the American Mathematical Society*, Vol. 21 (1920), p. 347.

components of M are a single ray and a bounded collection of arcs and points; and

(2) *No interior point of an arc- or ray-component, t, is a limit point of $M - t$.*

The conditions are evidently necessary. Their sufficiency is proved as follows: Suppose, first, that 1(a) and 2 hold. Let $p \epsilon E_n - M$, and let $S = S(p, d)$ be such that $S \cdot M = 0$. If S is taken as the sphere of inversion, the set M maps into a set $M' \subset S$. By familiar properties of the transformation of inversion, the set $M' + p$ is closed and satisfies the Moore-Kline conditions. The point p is a point-component of $M' + p$. Thus, by Theorem 3 there is an arc which contains $M' + p$, lies in S, and has p as one of its end points. The transform of this arc is a ray which contains M.

If 1(b) and 2 hold, $M' + p$ is again a set satisfying the Moore-Kline conditions. In this case p is an end point of an arc-component t of $M' + p$ and is not a limit point of $(M' + p) - t$. Again, Theorem 3 gives us the existence of an arc in S which contains $M' + p$ and has p as one end point. The transform of this arc is a ray which contains M.

It is here to be noted that G. T. Whyburn has given conditions which are necessary and sufficient that a closed set in E_2 be a subset of an open curve.* Whyburn's proof depends upon the Moore-Kline theorem. Using Theorem 1 of this paper in place of the Moore-Kline theorem, the result can be extended to $E_n (n > 2)$.

II.

In the first part of this section it will be shown that the Moore-Kline conditions constitute a solution of the problem of this paper only for a rather limited class of continuous curves.

THEOREM 5. *If S is a plane closed set, and if every one of its bounded closed subsets which satisfies the Moore-Kline conditions is a subset of an arc in S, then S is either the whole plane or a simple continuous curve.*

Proof. To abbreviate we will use the expression "a set E" to mean any bounded closed subset of S which satisfies the Moore-Kline conditions.

Since any pair of points of S is a set E, S is arc-wise connected and therefore connected. Since S is closed, it is a continuum. S is, in fact, a continuous curve. For if S is not a continuous curve, then in virtue of a

* *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 746-754; Theorem 5.

theorem of R. L. Wilder's * there is a point p of S and a circular neighborhood K of p such that p is a sequential limit point of a set of points $[p_n]$ of S each of which lies in a different quasi-component of $S \cdot K$. Evidently, $p + \sum p_n$ is a set E, and if t were an arc of S containing it, there would be in $S \cdot K$ a sub-arc t' of t such that $t' \supset p + \sum_{n=k}^{\infty} p_n (k \geqq 1)$. This, of course, contradicts the fact that no two points p_k and p_j lie in the same component of $S \cdot K$, and consequently S is a continuous curve.

It will now be shown that S is either the whole plane or else a simple continuous curve. Let us suppose, first, that S contains no simple closed curve. If S contained a point p which separated S into three mutually separated sets, S_1, S_2, and S_3, then if $p_i \epsilon S_i$ $(i = 1, 2, 3)$, the set $\sum_{i=1}^{3} p_i$ would be a set E which is clearly not a subset of any arc in S. Hence, no point p of S separates S into more than two components.

If S is bounded, it has at least two non-cut points, q_1 and q_2.† Let t be an arc of S from q_1 to q_2. If $t \neq S$, there is a component C of $S - t$ which has—since S is an acyclic continuous curve—a unique limit point q_3 in t. The set $S - q_3$, as is easily shown, would consist of three mutually separated sets. We have seen that this is impossible. Hence, if S is bounded, it is an arc.

Let us now suppose that S is unbounded and that it contains a non-cut point, p. The point p is the end point of a ray r of S.‡ If $r \neq S$, the unique limit point in r of a component C of $S - r$ would break up S into three mutually separated sets. Hence if S is unbounded and contains a non-cut point, it is a ray.

Let us suppose, finally, that S is unbounded and is disconnected by each one of its points. Thus if $p \epsilon S$, the set $S - p$ consists of two mutually separated connected sets, S_1 and S_2. The point p is a non-cut point of $S_i + p$ $(i = 1, 2)$, and by the argument just employed $S_i + p$ is a ray r_i, so that $S = r_1 + r_2$ is an open curve.

Now, let us suppose that S does contain a simple closed curve, J. Then S is cyclicly connected. For, suppose not. Then J determines a maximal

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* *Proceedings of the National Academy of Sciences*, Vol. 15 (1929), pp. 614-621; Theorem 1.

† S. Mazurkiewicz, *Fundamenta Mathematicae*, Vol. 2 (1921), pp. 119-130.

‡ C. Kuratowski, *Fundamenta Mathematicae*, Vol. 3 (1922), pp. 59-64.

cyclic curve * N of S such that $N \neq S$. If C is any component of $S - N$ then C has just one limit point, x, in N.* If a and b are two points of N, then, since N is cyclicly connected, there is an arc t in N from a to b one of whose interior points is x.† If $y \epsilon C$, the set $t + y$ is a set E, but it not a subset of any arc in S. Hence S is cyclicly connected.

Since S is a plane cyclicly connected continuous curve, the boundary of any domain complementary to S is either a simple closed curve or an open curve.† If $S \neq E_2$, there is at least one domain complementary to S.

Case 1. There is a domain D complementary to S whose boundary is a simple closed curve, J, of S.

It is easily shown that D is either the exterior or the interior of J. Let us suppose that D is the interior of J. (The argument is essentially the same if D is the exterior of J.) If $J \neq S$, there is a component C of $S - J$ which lies in the exterior of J. Since S is cyclicly connected, C has at least two limit points in J. Using the Wilder accessibility theorem, it follows that there exists an arc α whose end points a_1 and a_2 lie on J and which is such that $\alpha - a_1 - a_2 \subset C$. Let $a_1 a_2$ be that one of the two arcs of J from a_1 to a_2 which forms with α a simple closed curve whose interior lies exterior to J. Now let $r \epsilon J - a_1 a_2$ and let a'_1 and a'_2 be points of $a_1 a_2$ such that $a_1 < a'_1 < a'_2 < a_2$. Then the E-set $\alpha + a_1 a'_1 + a_2 a'_2 + r$ is clearly a subset of no arc in S. Thus $J = S$.

Case 2. There is no domain complementary to S whose boundary is a simple closed curve. We shall show that, under our supposition that S contains a simple closed curve, this case is impossible.

Let D be any domain complementary to S. The boundary, B, of D is an open curve. Since S contains a simple closed curve, $B \neq S$. There is, then, a component C of $S - B$. In the same way as in Case 1, we prove the existence of an arc α whose end points, a_1 and a_2, are points of B, and which is such that $\alpha - a_1 - a_2 \subset S - B$. Let $r \epsilon B - a_1 a_2$ and let a'_1 and a'_2 be points of $a_1 a_2$ such that $a_1 < a'_1 < a'_2 < a_2$. As D must be identical with that domain of the plane complementary to B which does not contain $\alpha - a_1 - a_2$, it is clear that the E-set $\alpha + a_1 a'_1 + a'_2 a_2 + r$ is a subset of no arc in S. Thus, Case 2 has been shown to be impossible.

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* G. T. Whyburn, *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 31-38.

† W. L. Ayres, *Bulletin de l'Académie Polonaise des Sciences et des Lettres* (1928); Theorem 3.

THEOREM 6. *If S is a closed set in a space E_n, and is of dimension 1 in at least one of its points, and if every bounded closed subset of S which satisfies the Moore-Kline conditions is a subset of an arc in S, then S is a simple continuous curve.*

Proof. As in the proof of the previous theorem, let us use the expression "a set E" to mean any bounded closed subset of S which satisfies the Moore-Kline conditions. As established in the previous theorem, S is a continuous curve, and if S contains no simple closed curve it is either an arc, a ray, or an open curve; and if S does contain a simple closed curve it is cyclicly connected.

We suppose, then, that S is cyclicly connected, and denote by p a point in which S is 1-dimensional. The point p lies on some simple closed curve J of S. Now if $J \neq S$, there is a component C of $S - J$. The component C has at least two limit points in J, since S is cyclicly connected. It follows that there is an arc in S which is a subset of $S - J$ except for one of its end points q which is a point of J distinct from p. Let Σ be a sphere whose center is p and which is so small that q does not lie in Σ or on its boundary. Since S is 1-dimensional in p there is a vicinity V of p such that $V \subset \Sigma$ and such that B, the boundary of V in S, is totally disconnected. As $B + p$ is a closed totally disconnected set and therefore a set E, it is a subset of an arc A of S. Now A cannot contain all the points of S outside V, for the point q lies outside V and is the junction point of three arcs which lie outside V. Therefore, there is a point h outside V which is not a point of A. If C_h is the component of $S - A$ which h determines, then C_h cannot have p as a limit point since p lies inside V, and h outside, and A contains the boundary B of V in S.

Now, there must be an arc in C_h from h to one of the end points a_1 or a_2, say, a_1—of A, for otherwise the E-set $A + h$ is a subset of no arc in S. Since S is cyclicly connected, C_h has another limit point in A. Let r be the last limit point of C_h on A in the order from a_1 to a_2. The point r is arc-wise accessible from C_h. For let a_1r be the arc of A from a_1 to r, and let $\{r_n\}$ be a sequence of points in C_h such that $\lim r_n = r$. The set $a_1r + [r_n]$ is an E-set and therefore a subset of an arc in S. Clearly, this arc contains an arc from a point r_n to r which lies in C_h except for r. Thus r is accessible from C_h. Let t_1 and t_r be mutually exclusive arcs of S, where a_1 is an end point of t_1, and r is an end point of t_r, and where $(t_1 - a_1) + (t_r - r) \subset C_h$. The point r must be a_2, for suppose not. Then the point set $t_1 + a_1r + t_r + a_2$ is a set E, but is clearly not a subset of any arc in S. Hence $r = a_2$.

If there were a component C_w of $S - A$ distinct from C_h, the point set $t_1 + a_1 r + t_r + w$ would be an E-set which is clearly not a subset of any arc in S. Thus C_h is the only component of $S - A$, so that $S = A + C_h$. Since a_1 and a_2 are both accessible from C_h, there is an arc L from a_1 to a_2 such that $L - a_1 - a_2 \subset C_h$. Now, p is an interior point of A and is not a limit point of C_h. Therefore there is a whole arc $x_1 p x_2$ of A, no point of which is a limit point of C_h, and where the order on A of the points a_1, x_1, p, x_2 and a_2 is a_1, x_1, p, x_2, a_2. Let $a_1 x_1$ and $x_2 a_2$ be the arcs of A from a_1 to x_1 and from x_2 to a_2, respectively. Suppose now that $C_h - C_h \cdot L \neq 0$ and let $s \epsilon C_h - C_h \cdot L$. The point set $a_1 x_1 + L + a_2 x_2 + s$ is a set E, but is a subset of no arc in S. Hence $C_h = C_h \cdot L$ so that S is the simple closed curve $L + A$. We have shown, then, that if S contains a simple closed curve, S is a simple closed curve. This completes the proof of the theorem.

It is not apparent that in three dimensions the closed sets for which the Moore-Kline theorem is true fall into any easily classified category. For instance, besides the topological plane and sphere, and the simple continuous curves, the theorem is true in three dimensions for such sets as the cube together with its interior, any closed two-dimensional manifold, and various continua made up of combinations of closed manifolds.

We shall now consider the general problem of determining when a given closed subset M of a continuous curve S lies on an arc of S.

THEOREM 7. *If S is a bounded continuous curve, and M a closed subset of S which is not a subset of any one maximal cyclic curve of S, then M is a subset of an arc in S if and only if the following three conditions are satisfied:*

(1) *If C is a cyclic element* of S, at most two components of $S - C$ contain a point of M.*

(2) *If C is a maximal cyclic curve of S, and K_1 and K_2 are distinct components of $S - C$ such that $K_i \cdot M \neq 0$ $(i = 1, 2)$, and such that a_1, the unique limit point of K_1 in C is distinct from a_2, the unique limit point of K_2 in C, then there is an arc t in C from a_1 to a_2 such that $M \cdot C \subset t$.*

(3) *If C is a maximal cyclic curve of S, and K is the only component of $S - C$ such that $K \cdot M \neq 0$, and a is the unique limit point of K in C, then there exists an arc t in C with a as one end point and such that $C \cdot M \subset t$.*

Proof. The conditions are sufficient.

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* See G. T. Whyburn, *American Journal of Mathematics*, Vol. 50 (1928), pp. 167-194.

Let W be the arc-curve $S(M)$.* The set W is a continuous curve* consisting of a certain collection of cyclic elements of S, and the cyclic elements of W are cyclic elements of S.† It will first be shown that if C is a cyclic element of W, then $W - C$ contains at most two components. Let C be any cyclic element of W. If K is a component of $W - C$, then $K \cdot M \neq 0$. For suppose the contrary and let a be the unique limit point of K in C, and let $x \epsilon K$. Since $K \subset W$, there are two points m_1 and m_2 of M which are the end points of an arc t in W which has x as an interior point. Not both of the sub-arcs m_1x and xm_2 of t can contain the point a. But m_1 and m_2 are both points of $W - K$ and a separates K from $W - K - a$. From this contradiction it follows that $K \cdot M \neq 0$.

Suppose, now, that $W - C$ contains three components K_i $(i = 1, 2, 3)$. Each component K_i is a subset of a component K'_i of $S - C$. In virtue of (1), two of these components are identical. We may suppose $K'_1 = K'_2$. Let $x \epsilon K_1 \cdot M$ and $y \epsilon K_2 \cdot M$. Now, $x + y \subset K'_1$, and there is an arc t in K'_1 from x to y.‡ Since $t \subset W$ and $K_1 + t + K_2$ is a connected subset of $W - C$, K_1 and K_2 lie in one component of $W - C$. As this contradicts the definition of K_1 and K_2, it follows that $W - C$ contains at most two components.

G. T. Whyburn has shown § that any bounded continuous curve, H, which is not cyclicly connected contains at least two nodes, a node being either an end point of H or a maximal cyclic curve N of H such that $H - I(N)$ is connected, $I(N)$ being the set of all points of N which are not limit points of $H - N$.

Since, by our hypothesis, M is not a subset of any one maximal cyclic curve of S, and since the cyclic elements of W are all cyclic elements of S, W cannot be cyclicly connected. Accordingly let N be a node of W. Suppose $W - N$ is not connected. Then N is evidently not a point since no end point of a continuous curve cuts the continuous curve. We have $W - N = W_1 + W_2$, where W_1 and W_2 are mutually separated. As we have already shown, W_1 and W_2 are connected. The set W_1 has a unique limit point a_1 and W_2 a unique limit point a_2 in N. Now $a_1 \neq a_2$, for if $a_1 = a_2$, the set $W - a_1$ contains three components $N - a_1$, W_1 and W_2. Evidently $W - I(N)$

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* See W. L. Ayres, *Transactions of the American Mathematical Society*, Vol. 30 (1928), pp. 567-578.

† W. L. Ayres, *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 595-612.

‡ See, for instance, R. L. Moore, *Bulletin of the American Mathematical Society*, Vol. 23 (1917), pp. 233-236.

§ See G. T. Whyburn, *American Journal of Mathematics, loc. cit.*

$= (W_1 + a_1) + (W_2 + a_2)$. Since W_1 and W_2 are mutually separated and since $a_1 \cdot \bar{W}_2 = a_2 \cdot \bar{W}_1 = 0$ it follows that $(W_1 + a_1)$ and $(W_2 + a_2)$ are mutually separated. As this contradicts the fact that N is a node, every node of W must be a non-cut element of W, so that W contains at least two non-cut elements N_1 and N_2.

Let X be any simple cyclic chain * in W from N_1 to N_2. It will be shown that $X = W$. Suppose, in fact, $X \neq W$, and let $p \epsilon W - X$. The point p determines a component C_p of $W - X$ which has a unique limit point q in X. Since N_1 and N_2 are non-cut elements of W, they are, in virtue of a theorem of Whyburn's,† end elements of W. In other words, q is not a point of either N_1 or N_2. Hence, q is a point of some cyclic element Q of X interior to X. Clearly, $W - Q$ contains three components determined by $N_1 - N_1 \cdot Q$, $N_2 - N_2 \cdot Q$, and C_p. As we have shown that this is impossible, it follows that W is the simple cyclic chain X.

Now, let t be an arc in W from a point of N_1 to a point of N_2. In virtue of a theorem of Whyburn's,‡ the arc t contains a point of every cyclic element of W, and in fact, if C is a maximal cyclic curve of W interior to X, then $t \cdot C$ is a sub-arc of t whose end points are the unique limit points in C of the two distinct components of $W - C$ and therefore of $S - C$. Let us now replace all such sub-arcs of t by the corresponding arcs whose existence is given by (2), and let us replace $t \cdot N_1$ and $t \cdot N_2$ by sub-arcs of N_1 and N_2 of the sort whose existence is given by (3). Since for every $\epsilon > 0$ there are only a finite number of maximal cyclic curves of W of diameter $> \epsilon$ it is readily shown that the resulting point set is an arc, A. As $M \subset A \subset W \subset S$, the conditions have been shown to be sufficient.

The conditions are also necessary.

Let t be an arc of S such that $t \supset M$, and let us suppose that there is a cyclic element C of S such that three components, S_1, S_2, and S_3 of $S - C$ each contain a point of M. Let $p_i \epsilon M \cdot S_i$, and let us suppose that the order of the points p_1, p_2, and p_3 on t is p_1, p_2, p_3. Since these three points lie in different components of $S - C$, the sub-arcs p_1p_2 and p_2p_3 of t each contain a point of C. Evidently, then, S_2 has at least two limit points in C. As this is impossible, the necessity of (1) is proved.

To prove the necessity of (2) let C be any maximal cyclic curve of S and suppose there exist components K_1 and K_2 of $S - C$ and points a_1 and a_2

* See G. T. Whyburn, *loc. cit.*
† *Loc. cit.*
‡ *Loc. cit.*

as specified in (2). Since each point a_1 and a_2 separates K_1 from K_2, t must contain a_1 and a_2. The sub-arc a_1a_2 of t is a subset of C, for otherwise there is determined a component of $S - C$ which has at least two limit points in C. If $(t - a_1a_2) \cdot C \neq 0$ either K_1 or K_2 would have two limit points in C. Hence $a_1a_2 = t \cdot C$. Since $t \supset M \supset C \cdot M$, the arc a_1a_2 of C must contain $C \cdot M$. The necessity of (3) follows in a similar way. This completes the proof of the theorem.

We shall now make certain applications of the theorem just proved.

THEOREM 8. *If the plane continuous curve S is the boundary of a (connected) domain D and M is any closed subset of S other than a simple closed curve, then M is a subset of an arc in S if and only if the following three conditions are satisfied:*

(1) *If F is a point or a simple closed curve of S, at most two components of $S - F$ contain a point of M.*

(2) *If J is a simple closed curve in S, and if there are two components of $S - J$ which contain a point of M and have distinct limit points a_1 and a_2 in J, then only one of the arcs of J from a_1 to a_2 contains a point of M.*

(3) *If J is a simple closed curve in S, and if just one component of $S - J$ contains a point of M, then the (unique) limit point in J of this component is not a limit point of $J \cdot M$ from both directions along J.**

Proof. The theorem is almost an immediate consequence of Theorem 7 and the fact that every maximal cyclic curve of a continuous curve which is the boundary of a plane domain is a simple closed curve.† It is clear that the above theorem is true for any continuous curve in E_n whose maximal cyclic curves are all simple closed curves.

THEOREM 9. *If S is an acyclic continuous curve in E_n and M is a closed subset of S, then in order that M be a subset of an arc in S it is necessary and sufficient that if p is a point of S, at most two components of $S - p$ contain a point of M.*

Proof. This theorem is an immediate consequence of Theorem 7 and the fact that every cyclic element of an acyclic continuous curve is a point.

Theorem 9 can evidently be made more general by letting S be *any continuous curve in E_n, M any closed subset of S which has no point on any simple closed curve of S, and p any cyclic element of S.*

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* The phrase "from both directions along J" is immediately translatable into topological language.

† R. L. Wilder, *Fundamenta Mathematicae*, Vol. 7 (1923), pp. 340-377.

With a view to making another application of Theorem 7, the following theorem will be proved:

THEOREM 10. *If J is a simple closed curve in E_2, and I is its interior, and if F is a closed totally disconnected subset of $J + I$, then there is an arc in $J + I$ which contains F.*

Proof. Let us denote $F \cdot J$ by T. There is an arc AB of J which has T in its interior. Let C and D be two other points of J and let the order of the points A, B, C, and D on J be A, B, C, D, A, where T is contained in the arc ABC of J. Let a system of rectangular coördinates be given, and let A', B', C' and D' be the points $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, respectively. Let J' be the simple closed curve formed by the square $A'B'C'D'A'$, and let I' be its interior. There will be defined a $1 - 1$ bicontinuous mapping of $J + I$ into $J' + I'$ which has the properties:

(1) J corresponds to J'; the points of AB corresponding to the points of $A'B'$, A corresponding to A' and B to B'.

(2) The points of T correspond to irrational points of $A'B'$.

(3) The points of $F - T$ correspond to points of I' both of whose coördinates are irrational.

The proof that such a mapping exists is based on a lemma in a paper of R. L. Wilder's,* and for the meaning of the notation used in the next paragraph, the reader is referred to the paper in question.

There is a $1 - 1$ bicontinuous correspondence between the points of AB and the points of $A'B'$ in which A corresponds to A' and B to B', the rational points within $A'B'$ corresponding to points of AB which are end points of the arcs a_1, and the irrational points of $A'B'$ corresponding to points of AB which are end points of the arcs b_1. There is a like $1 - 1$ bicontinuous correspondence between the points of $A'D'$ and AD. These two correspondences induce like correspondences between the points of BC and $B'C'$, CD and $C'D'$. If $p \epsilon I$, there is just one arc t_1 of α_1, and just one arc t_2 of α_2 which contains p. That end point of t_1 which belongs to AB corresponds to a point x of $A'B'$, and that end point of t_2 which belongs to AD corresponds to a point y of $A'D'$. To the point p we let correspond the point (x, y) of I'. It is readily shown that the correspondence in question is $1 - 1$ bicontinuous and has the properties (1), (2), and (3) specified above.

Consider, now, the square J', and let F' and T' be the sets corresponding

* *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 345-359; Lemma 6, p. 356.

to F and T. The set T' is a closed non-dense subset of $A'B'$. Let R_1 be the rectangle whose vertices are the points $(0, \frac{1}{2})$, $(1, \frac{1}{2})$, $(1, 1)$ and $(0, 1)$. The projection, p_1, on $A'B'$ of the mid-point of the base of R_1 is the point $(\frac{1}{2}, 0)$. Since p_1 is a rational point, it belongs to some sub-interval of $A'B'$ complementary to T'. Let $(a_1, 0)$ and $(b_1, 0)$ be any two rational points of this interval such that $a_1 <$ absc. of $p_1 < b_1$. Let J_1 be the simple closed curve obtained by deleting from R_1 the sub-interval of R_1 between $(a_1, \frac{1}{2})$ and $(b_1, \frac{1}{2})$ and adding the lines representing the ordinates of these two points and the interval, i_1, of $A'B'$ from $(a_1, 0)$ to $(b_1, 0)$. Denote, now, by R_2 the rectangle whose vertices are the points $(0, \frac{1}{2^2})$, $(a_1, \frac{1}{2^2})$, $(a_1, \frac{1}{2})$ and $(0, \frac{1}{2})$. The projection, p_2, on $A'B'$ of the mid-point of the base of R_2 is a rational point and therefore belongs to some sub-interval of $A'B'$ complementary to T'. Let $(a_2, 0)$ and $(b_2, 0)$ be any two rational points of this interval such that $a_2 <$ absc. of $p_2 < b_2$, and let us denote the interval of $A'B'$ from $(a_2, 0)$ to $(b_2, 0)$ by i_2. These points are related to R_2 in the same way that $(a_1, 0)$ and $(b_1, 0)$ are related to R_1. Let us, then, construct J_2 in relation to them and to R_2 in the same way that J_1 was constructed in relation to $(a_1, 0)$ and $(b_1, 0)$ and R_1. We now consider the rectangle R_3 whose vertices are the points $(b_1, \frac{1}{2^2})$, $(1, \frac{1}{2^2})$, $(1, \frac{1}{2})$ and $(b_1, \frac{1}{2})$. We define in a similar way a simple closed curve J_3. At the next stage of the process we have 2^2 rectangles to consider, then 2^3 rectangles, and so on.

Proceeding indefinitely in the manner indicated, it is clear that we obtain a sequence $\{J_n\}$ of simple closed curves such that:

(1) $J_n \cdot F' = 0 \qquad (n = 1, 2, \cdots)$.

(2) If H_n is the interior of J_n, then $H_i \cdot H_j = 0$ $(i \neq j)$ $(i, j = 1, 2, \cdots)$.

(3) $F' - T' \subset \sum_{n=1}^{\infty} H_n$.

(4) $\lim \operatorname{diam}(J_n) = 0$.

The set F'_n is a closed totally disconnected set. There is, then, in virtue of Theorem 2 of this paper an arc t_n in H_n such that $t_n \supset F'_n$. It is easy to show that the end points of this arc can be joined by arcs to a_n and b_n so as to give an arc t'_n such that $t'_n \supset F'_n$ and $t'_n - a_n - b_n \subset H_n$. It follows easily that the point set $\sum_{n=1}^{\infty} t'_n + (A'B' - \sum_{n=1}^{\infty} i_n)$ is an arc, t', whose end points are A' and B'. Evidently $F' \subset t' \subset J' + I'$. In virtue of the homeomorphism between $J + I$ and $J' + I'$, the theorem is established.

THEOREM 11. *Under the conditions of Theorem* 10, *if a and b are two*

points of F, then there is an arc in $J + I$ which contains F and has a and b as its end points.

The proof is based on Theorem 10 and uses the homeomorphism between $J + I$ and $J' + I'$ set up in the proof of that theorem.

It is possible to show that the arc of Theorem 11 can be so constructed that the only points it has in common with J are the points of F in J.

We shall now use Theorem 11 to make another application of Theorem 7.

THEOREM 12. *If S is a bounded plane continuous curve which does not cut the plane, and F is a closed totally disconnected subset of S, then in order that F lie on an arc of S, it is necessary and sufficient that for every cyclic element C of S at most two components of $S - C$ contain a point of F.*

Proof. Every maximal cyclic curve of S is a simple closed curve plus its interior.* If C is such that there are two points a_1 and a_2 of the sort specified under condition (2), Theorem 7, then in virtue of Theorem 11, there is an arc in C which contains the closed totally disconnected set $C \cdot F + a_1 + a_2$ and has a_1 and a_2 as its end points. Similarly, if condition (3), Theorem 7, applies, there is an arc in C which contains $C \cdot F + a$, and has a as one of its end points. Thus the conditions of Theorem 7 are all satisfied and Theorem 12 follows.

As a special case of the general problem of this paper it is of interest to consider the problem of determining necessary and sufficient conditions that a given finite subset of a continuous curve S lie on an arc of S. A condition which is sufficient but not necessary has been given by W. L. Ayres.† Evidently, if S is any set of points whatsoever and M is a subset of S consisting of just n points, then for M to lie on an arc of S it is *necessary* that if $p_1, \cdots p_i$ are any i points of S $(i < n-1)$ at most $i+1$ components of $S - (p_1 + \cdots + p_i)$ contain a point of M. This condition is clearly not sufficient for continuous curves in general.

We have found conditions which are at the same time necessary and sufficient that a given set of n points of a continuous curve S lie on an arc of S for the cases $n = 3$ and $n = 4$. These are embodied in the following two theorems whose proofs we do not feel it will be necessary to include.

THEOREM 13. *If S is a continuous curve in E_n and M is a subset of S consisting of just three points, then M is a subset of an arc in S if and only*

* See G. T. Whyburn, *American Journal of Mathematics, loc. cit.*

† This paper is not yet published. For abstract, see *Bulletin of the American Mathematical Society*, Vol. 35 (1929), p. 772.

if M contains a point which is not separated from both of the other points of M by any one point of S.

THEOREM 14.* *If S is a cyclicly connected continuous curve in E_n and M consists of four points of S, then M is a subset of an arc in S if and only if for every pair of points p_1 and p_2 of S at most three components of $S - (p_1 + p_2)$ contain a point of M.*

Simple examples show that none of the evident generalizations of the conditions in Theorems 13 and 14 apply to the general case of *any* finite subset of a continuous curve.

* This theorem is a generalization of a theorem of W. L. Ayres, *Bulletin de l'Académie Polonaise des Sciences et des Lettres* (1928), Theorem 5.

NOTE ON THE CONDITION THAT A BOOLEAN EQUATION HAVE A UNIQUE SOLUTION.*

By B. A. BERNSTEIN.

Consider the general Boolean equation

(i) $f(x, y, \cdots, t) \equiv Axy \cdots t + Bxy \cdots t' + \cdots + Lx'y' \cdots t' = 0$,†

involving n unknowns $x, y, \cdots, t$ and having $A, B, \cdots, L$ for its 2^n discriminants. Consider also the equations

(ii) $$x = a, \quad y = b, \cdots, \quad t = k,$$

and the equation

(iii) $\phi(x, y, \cdots, t) \equiv a'x + ax' + b'y + by' + \cdots + k't + kt' = 0.$

Whitehead ‡ has shown that the necessary and sufficient condition that (i) have the unique solution (ii) is that (i) be of the form (iii). Professor Whitehead's proof, carried out only for an equation in two unknowns, consists in showing that the necessary and sufficient condition that (i) have the unique solution (ii) is relation

(iv) $$AB \cdots L + A'B' + A'C' + \cdots + K'L' = 0,$$

which relation he proved earlier to be a necessary and sufficient condition that f of (i) be of the form ϕ of (iii). But Professor Whitehead's proofs require a good bit of calculation, especially for the general case of n variables.§ The object of this note is: (1) to offer a very simple proof of the fact that an equation of form (iii) has the solution (ii) and conversely; (2) to offer a very simple proof of the fact that (iv) is the condition that f of (i) is of the form ϕ of (iii); and (3) to call attention to the simple geometry underlying condition (iv).

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* Presented to the Society, April 11, 1931.

† The usual Boole-Schröder notation is used, except that a' denotes the negative of a.

‡ A. N. Whitehead, "Memoir on the Algebra of Symbolic Logic," *American Journal of Mathematics*, Vol. 23 (1901), pp. 140-150.

§ To obtain (iv) as the necessary condition that (i) have a unique solution Whitehead eliminates from (i) all the variables except one, in turn, then gets the conditions that the resulting equations have unique solutions, and then combines these conditions into a single condition. To obtain (iv) as the condition that f of (i) be of the form ϕ of (iii), Professor Whitehead develops f normally with respect to the variables, then identifies corresponding discriminants, and then simplifies.

1. A very simple proof of the fact that an equation of form (iii) has the solution (ii) and conversely, consists in merely observing that equation (iii) and the system of equations (ii) are each equivalent to the system

(v) $$a'x + ax' = 0, \quad b'y + by' = 0, \cdots, \quad k't + kt' = 0.$$

It will be noted that this proof does not use (iv) at all.

2. To obtain (iv) very simply as the necessary and sufficient condition that f of (i) be of the form ϕ of (iii), observe that the discriminants of ϕ are

$$\phi(1, 1, \cdots, 1) = a' + b' + \cdots + k', \quad \phi(1, 1, \cdots, 0) = a' + b' + \cdots + k,$$
$$\phi(0, 0, \cdots, 0) = a + b + \cdots + k.$$

Hence, f will be identical with ϕ when and only when

$$A = a' + b' + \cdots + k', \quad B = a' + b' + \cdots + k, \cdots, \quad L = a + b + \cdots + k,$$

or

(vi) $$A' = ab \cdots k, \quad B' = ab \cdots k', \cdots, \quad L' = a'b' \cdots k'.$$

But the right-hand members of (vi) are seen to be the *constituents* in the normal development of 1 with respect to $a, b, \cdots, k$. Hence,

$$A' + B' + \cdots + L' = 1, \quad A'B' = 0, \quad A'C' = 0, \cdots, \quad K'L' = 0,$$

or

(iv) $$AB \cdots L + A'B' + A'C' + \cdots + K'L' = 0.$$

Of course, since (ii) is equivalent to (iii), we have that (iv) is also the condition that (i) have the solution (ii).

3. The simple geometry underlying condition (iv) is seen from (vi). The necessary and sufficient condition that (i) have the solution (ii), or that f of (i) be of the form ϕ of (iii), is, geometrically, that the regions representing the negatives of the discriminants of f be the regions corresponding to the 2^n constituents in the normal development of 1 with respect to $a, b, \cdots, k$ of ϕ. The geometry for the general equation (i) is thus a very simple extension of the geometry for the equation in one unknown, $a'x + ax' = 0$.

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ON THE EQUIVALENCE OF THE DECOMPOSITIONS OF AN ALGEBRA WITH RESPECT TO A PRINCIPAL IDEMPOTENT.

By LAURENS EARLE BUSH.

1. *Introduction.* In the thesis of the author entitled *Some Properties of Algebras without Moduli* * the results obtained were apparently dependent upon the decomposition of the algebra with respect to a definitely selected principal idempotent. This naturally raises a question as to the extent to which this dependence is actual. The purpose of this paper is, at least partially, to answer this question.

2. *Decomposition of an algebra with respect to a principal idempotent.* Let $\mathfrak{A}$ be a non-nilpotent algebra over an infinite field $\mathfrak{F}$. $\mathfrak{A}$ possesses at least one principal idempotent element u.† Let $\mathfrak{B}$ be the set of all elements x of $\mathfrak{A}$ such that $uxu = 0$, and $\mathfrak{f}$ the set of all elements x of $\mathfrak{A}$ such that $ux = xu = 0$. Then

$$\mathfrak{A} = u\mathfrak{A}u + u\mathfrak{B} + \mathfrak{B}u + \mathfrak{f},‡$$

where the linear systems $u\mathfrak{A}u$, $u\mathfrak{B}$, $\mathfrak{B}u$, and $\mathfrak{f}$ are supplementary in $\mathfrak{A}$. We have further that $u\mathfrak{A}u = \mathfrak{S} + \bar{\mathfrak{N}}$,§ where $\mathfrak{S}$ is a semi-simple subalgebra of $\mathfrak{A}$, not a zero algebra of order one, having the modulus u and $\bar{\mathfrak{N}}$ is the radical (maximal nilpotent invariant subalgebra) of $u\mathfrak{A}u$, and

$$\bar{\mathfrak{N}} + u\mathfrak{B} + \mathfrak{B}u + \mathfrak{f} = \mathfrak{N},$$

where $\mathfrak{N}$ is the radical of $\mathfrak{A}$.

It is easily shown that the elements of the systems $\mathfrak{S}$, $\bar{\mathfrak{N}}$, $u\mathfrak{B}$, $\mathfrak{B}u$, and $\mathfrak{f}$ form products according to the following table.

	$\mathfrak{S}$	$\bar{\mathfrak{N}}$	$u\mathfrak{B}$	$\mathfrak{B}u$	$\mathfrak{f}$
$\mathfrak{S}$	$\mathfrak{S}$	$\bar{\mathfrak{N}}$	$u\mathfrak{B}$	0	0
$\bar{\mathfrak{N}}$	$\bar{\mathfrak{N}}$	$\bar{\mathfrak{N}}$	$u\mathfrak{B}$	0	0
$u\mathfrak{B}$	0	0	0	$\bar{\mathfrak{N}}$	$u\mathfrak{B}$
$\mathfrak{B}u$	$\mathfrak{B}u$	$\mathfrak{B}u$	$\mathfrak{f}$	0	0
$\mathfrak{f}$	0	0	0	$\mathfrak{B}u$	$\mathfrak{f}$

* *The Ohio State University, Abstracts of Doctors' Dissertations*, No. 7 (1931).

† L. E. Dickson, *Algebren und ihre Zahlentheorie*, Zurich, 1927, p. 100.

‡ J. H. M. Wedderburn, *Proceedings of the London Mathematical Society*, Vol. 6, 2nd series (1907-8), pp. 91-93, and Dickson, *loc. cit.*, pp. 98-99.

§ Dickson, *loc. cit.*, p. 136.

For example, if x and y are respectively in $u\mathfrak{B}$ and $\mathfrak{B}u$, xy is in $\bar{\mathfrak{N}}$, or if x and y are respectively in $\bar{\mathfrak{N}}$ and $\mathfrak{F}$, $xy = 0$.

3. *Some theorems on the principal idempotents of an algebra.* If u_1 and u_2 are principal idempotents of $\mathfrak{A}$, $u_1 - u_2$ is properly nilpotent in $\mathfrak{A}$.* We have conversely

THEOREM 3.1. *If $u' = u + n$ is idempotent, where u is a principal idempotent of $\mathfrak{A}$ and n is properly nilpotent in $\mathfrak{A}$, u' is a principal idempotent of $\mathfrak{A}$.*

For, let $e = x_1 + x_2$ be any idempotent, where x_1 is in that determination of $\mathfrak{S}$ for which u is a modulus and x_2 is in $\mathfrak{N}$. Then

$$u'e = (u + n)(x_1 + x_2) = x_1 + nx_1 + ux_2 + nx_2.$$

Since e is idempotent, $x_1 \neq 0$. But x_1 is in $\mathfrak{S}$, while, on account of the invariance of $\mathfrak{N}$, $nx_1 + ux_2 + nx_2$ is in $\mathfrak{N}$. Hence $u'e \neq 0$, and u' is a principal idempotent.

THEOREM 3.2. *If u is a principal idempotent of $\mathfrak{A}$ and x is any element of $u\mathfrak{B}$ or $\mathfrak{B}u$, $u + x$ is a principal idempotent of $\mathfrak{A}$.*

For, $(u + x)^2 = u + x$ and $u + x$ is a principal idempotent by theorem 3.1.

THEOREM 3.3. *If u is a principal idempotent of $\mathfrak{A}$ and $\mathfrak{B}u\mathfrak{B} = 0$, then $e = u - x_2x_3 + x_2 + x_3$, where x_2 is in $u\mathfrak{B}$ and x_3 in $\mathfrak{B}u$, is a principal idempotent of $\mathfrak{A}$, and every principal idempotent of $\mathfrak{A}$ is of this form.*

For, since $\mathfrak{B}u\mathfrak{B} = 0$, $x_3x_2 = 0$, and therefore $e^2 = e$. By theorem 3.1, e is a principal idempotent of $\mathfrak{A}$. Conversely, suppose e is any principal idempotent of $\mathfrak{A}$. Then $e - u$ is in $\mathfrak{N}$ by the remark at the beginning of this section, and we may write $e = u + x_1 + x_2 + x_3 + x_4$, where x_1, x_2, x_3, and x_4 are respectively in $\bar{\mathfrak{N}}$, $u\mathfrak{B}$, $\mathfrak{B}u$, and $\mathfrak{F}$. Since $e^2 = e$, we have

$$(x_1 + x_1^2 + x_2x_3) + (x_1x_2 + x_2x_4) + (x_3x_1 + x_4x_3) + (x_4^2 - x_4) = 0.$$

Since $\bar{\mathfrak{N}}$, $u\mathfrak{B}$, $\mathfrak{B}u$, and $\mathfrak{F}$ are supplementary in their sum, the terms in each of these systems must vanish separately, i. e.

$$(1) \qquad x_1 + x_1^2 + x_2x_3 = 0,$$
$$(2) \qquad x_1x_2 + x_2x_4 = 0,$$
$$(3) \qquad x_3x_1 + x_4x_3 = 0,$$
$$(4) \qquad x_4^2 - x_4 = 0.$$

* Wedderburn, *loc. cit.*, p. 92.

Since x_4 is nilpotent or zero, (4) requires that $x_4 = 0$. By (2), $x_1x_2 = 0$. Multiplying (1) on the left by x_1 and x_1^2 in turn and subtracting the resulting equations, we have, after transposing terms, $x_1^2 = x_1^4$. Since x_1^2 is nilpotent or zero, this requires that $x_1^2 = 0$. Then, by (1), $x_1 = -x_2x_3$. Hence $e = u - x_2x_3 + x_2 + x_3$.

In a similar manner we may prove

THEOREM 3.4. *If u is a principal idempotent of $\mathfrak{A}$ and $u\mathfrak{B}^2u = 0$ (i.e., if $\mathfrak{B}$ is an algebra), then $e = u + x_2 + x_3 + x_3x_2$, where x_2 is in $u\mathfrak{B}$ and x_3 in $\mathfrak{B}u$, is a principal idempotent of $\mathfrak{A}$, and every principal idempotent of $\mathfrak{A}$ is of this form.*

It is evident that $\mathfrak{A}$ and $\mathfrak{A}^r$ have the same set of idempotents, and therefore also of principal idempotents. If $\mathfrak{A}$ is of index r, $\mathfrak{A}$ therefore has a unique principal idempotent if and only if $\mathfrak{A}^r$ has. If $\mathfrak{A}^r$ has a modulus, it, and consequently $\mathfrak{A}$, has a unique principal idempotent.* If, conversely, $\mathfrak{A}$ has a unique principal idempotent u, by theorem 3.2, $u\mathfrak{B} = \mathfrak{B}u = 0$. Then $\mathfrak{A} = u\mathfrak{A}u + \mathfrak{F}$ and

$$\mathfrak{A}^r = u\mathfrak{A}u + \mathfrak{F}^r = \mathfrak{A}^{r+1} = u\mathfrak{A}u + \mathfrak{F}^{r+1}.$$

Hence, since $\mathfrak{F}$ is nilpotent and $\mathfrak{F}^r = \mathfrak{F}^{r+1}$, we have $\mathfrak{F}^r = \mathfrak{F}^{r+1} = 0$, and $\mathfrak{A}^r = u\mathfrak{A}u$ has the modulus u. Thus we have

COROLLARY 3.41. *A non-nilpotent algebra $\mathfrak{A}$ of index r has a unique principal idempotent if and only if $\mathfrak{A}^r$ has a modulus.*

4. *Equivalence of decompositions.* Let u_1 and u_2 be two different principal idempotents of $\mathfrak{A}$, and let the decompositions of $\mathfrak{A}$ with respect to u_1 and u_2 be respectively

$$\mathfrak{A} = \mathfrak{S}_1 + \bar{\mathfrak{N}}_1 + u_1\mathfrak{B}_1 + \mathfrak{B}_1u_1 + \mathfrak{F}_1,$$

and

$$\mathfrak{A} = \mathfrak{S}_2 + \bar{\mathfrak{N}}_2 + u_2\mathfrak{B}_2 + \mathfrak{B}_2u_2 + \mathfrak{F}_2.$$

These two decompositions will be called *equivalent* if it is possible to associate with every element x of $\mathfrak{A}$ a unique element $\bar{x}$ of $\mathfrak{A}$ in such a manner that

(1) Every element of $\mathfrak{A}$, considered as an $\bar{x}$, is associated with a unique element x;

(2) The association is isomorphic under addition, subtraction, and both types of multiplication;

(3) According as x is in $\mathfrak{S}_1$, $\bar{\mathfrak{N}}_1$, $u_1\mathfrak{B}_1$, $\mathfrak{B}_1u_1$, or $\mathfrak{F}_1$, the associated

* Dickson, *loc. cit.*, p. 100.

element $\bar{x}$ is in the corresponding system $\mathfrak{S}_2$, $\bar{\mathfrak{N}}_2$, $u_2\mathfrak{B}_2$, $\mathfrak{B}_2u_2$, or $\mathfrak{F}_2$ respectively.

It is evident from the definition that equivalence of decompositions is a transitive relation.

THEOREM 4.1. *A necessary and sufficient condition that the decompositions of $\mathfrak{A}$ with respect to u_1 and u_2 be equivalent is that it be possible to associate with every element x of $\mathfrak{A}$ a unique element $\bar{x}$ of $\mathfrak{A}$ in such a manner that* (1) *and* (2) *are satisfied, and that $\bar{u}_1 = u_2$.*

For, suppose (1), (2), and (3) are satisfied. If x is in $\mathfrak{S}_1$, $\bar{x}$ is in $\mathfrak{S}_2$. In particular, $\bar{u}_1$ is in $\mathfrak{S}_2$. For every x in $\mathfrak{S}_1$, $u_1x = x$ and $xu_1 = x$, and therefore for every $\bar{x}$ in $\mathfrak{S}_2$, $\bar{u}_1\bar{x} = \bar{x}$ and $\bar{x}\bar{u}_1 = \bar{x}$. That is $\bar{u}_1$ is a modulus for $\mathfrak{S}_2$, and since u_2 is the only modulus for $\mathfrak{S}_2$, $\bar{u}_1 = u_2$.

Conversely, suppose that (1) and (2) are satisfied and that $\bar{u}_1 = u_2$. Let $e_1 = u_1, e_2, e_3, \cdots, e_n$ be a basis for $\mathfrak{A}$, such that $e_1, e_2, \cdots, e_{\rho_1}$ is a basis for $\mathfrak{S}_1$, $e_{\rho_1+1}, e_{\rho_1+2}, \cdots, e_{\rho_2}$ a basis for $\bar{\mathfrak{N}}_1$, $e_{\rho_2+1}, e_{\rho_2+2}, \cdots, e_{\rho_3}$ a basis for $u_1\mathfrak{B}_1$, $e_{\rho_3+1}, e_{\rho_3+2}, \cdots, e_{\rho_4}$ a basis for $\mathfrak{B}_1u_1$, and $e_{\rho_4+1}, e_{\rho_4+2}, \cdots, e_n$ a basis for $\mathfrak{F}_1$. On account of (2), the elements $\bar{e}_1 = u_2, \bar{e}_2, \bar{e}_3, \cdots, \bar{e}_n$, which correspond to $e_1, e_2, \cdots, e_n$ respectively, are linearly independent with respect to $\mathfrak{F}$, and can therefore be selected as a new basis for $\mathfrak{A}$. The multiplication constants for the new basis will be identical with those for the old, and it follows that in the decomposition with respect to $u_2, \bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{\rho_1}$ will form a basis for some determination of $\mathfrak{S}_2$, $\bar{e}_{\rho_1+1}, \bar{e}_{\rho_1+2}, \cdots, \bar{e}_{\rho_2}$ a basis for $\bar{\mathfrak{N}}_2$, $\bar{e}_{\rho_2+1}, \bar{e}_{\rho_2+2}, \cdots, \bar{e}_{\rho_3}$ a basis for $u_2\mathfrak{B}_2$, $\bar{e}_{\rho_3+1}, \bar{e}_{\rho_3+2}, \cdots, \bar{e}_{\rho_4}$ a basis for $\mathfrak{B}_2u_2$, and $\bar{e}_{\rho_4+1}, \bar{e}_{\rho_4+2}, \cdots, \bar{e}_n$ a basis for $\mathfrak{F}_2$. Since if $x = \sum_{i=1}^{n} x_ie_i$, it follows that $\bar{x} = \sum_{i=1}^{n} x_i\bar{e}_i$, (3) is satisfied.

By theorem 3.2, if u is a principal idempotent of $\mathfrak{A}$ and x is in $u\mathfrak{B}$ or $\mathfrak{B}u$, then $u + x$ is a principal idempotent of $\mathfrak{A}$. We shall prove the following theorem regarding the set of principal idempotents obtained in this manner.

THEOREM 4.2. *If u is a principal idempotent of $\mathfrak{A}$ and x is in $u\mathfrak{B}$ or $\mathfrak{B}u$, the decomposition of $\mathfrak{A}$ with respect to $e = u + x$ is equivalent to the decomposition of $\mathfrak{A}$ with respect to u.*

In order to prove this we shall set up a correspondence satisfying the conditions of theorem 4.1. Let us suppose that x is in $u\mathfrak{B}$ and let $\mathfrak{A} = u\mathfrak{A}u + u\mathfrak{B} + \mathfrak{B}u + \mathfrak{F} = e\mathfrak{A}e + e\bar{\mathfrak{B}} + \bar{\mathfrak{B}}e + \mathfrak{F}$. Let $y = y_1 + y_2 + y_3 + y_4$, where y_1, y_2, y_3, and y_4 are respectively in $u\mathfrak{A}u$, $u\mathfrak{B}$, $\mathfrak{B}u$, and $\mathfrak{F}$, be any element of $\mathfrak{A}$. To y we let correspond the element

$$\bar{y} = y + y_1x + y_3x - xy_3 - xy_4 - xy_3x,$$

which is uniquely determined by y. Conversely, if $\bar{y} = \bar{y}_1 + \bar{y}_2 + \bar{y}_3 + \bar{y}_4$, where $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$, and $\bar{y}_4$ are respectively in $u\mathfrak{A}u$, $u\mathfrak{B}$, $\mathfrak{B}u$, and $\mathfrak{F}$, is any element of $\mathfrak{A}$, it is the correspondent by the above formula of the uniquely determined element $y = \bar{y} + x\bar{y}_3 + x\bar{y}_4 - \bar{y}_1 x - \bar{y}_3 x - x\bar{y}_3 x$. Direct computation shows that this correspondence is isomorphic under addition, subtraction, and both kinds of multiplication. Also $\bar{u} = u + ux = u + x = e$. Thus the conditions of theorem 4. 1 are satisfied. The proof is similar when x is in $\mathfrak{B}u$.

As an illustration of theorem 4. 2, consider the algebra defined by the following table.

	e_1	e_2	e_3	e_4	e_5
e_1	e_1	e_2	e_3	0	0
e_2	e_2	0	0	0	0
e_3	0	0	0	e_2	0
e_4	e_4	0	0	0	0
e_5	0	0	0	0	0

The element e_1 is a principal idempotent and in the decomposition with respect to it we have e_1, e_2, e_3, e_4, and e_5 respectively in $\mathfrak{S}$, $\bar{\mathfrak{N}}$, $u\mathfrak{B}$, $\mathfrak{B}u$, and $\mathfrak{F}$. The element $e_1 + e_3$ is a principal idempotent of the form required by theorem 4. 2. In the correspondence set up by the proof of this theorem, we have that e_1, e_2, e_3, e_4, and e_5 correspond respectively to $e_1 + e_3$, e_2, e_3, $e_4 - e_2$, and e_5. If we select the latter five elements as a new basis, calling them respectively e'_1, e'_2, e'_3, e'_4, and e'_5, we obtain a new multiplication table which is identical with the original one with e_i $(i = 1, 2, 3, 4, 5)$ replaced by e'_i.

THEOREM 4. 3. *If u is a principal idempotent of $\mathfrak{A}$ and $\mathfrak{B}u\mathfrak{B} = 0$, the decomposition of $\mathfrak{A}$ with respect to any principal idempotent is equivalent to that with respect to u.*

For, let e be any principal idempotent of $\mathfrak{A}$. By theorem 3. 3, $e = u - x_2 x_3 + x_2 + x_3$, where x_2 and x_3 are respectively in $u\mathfrak{B}$ and $\mathfrak{B}u$. By theorem 3. 2, $\bar{e} = u + x_2$ is a principal idempotent of $\mathfrak{A}$, and by theorem 4. 2, the decomposition of $\mathfrak{A}$ with respect to $\bar{e}$ is equivalent to that with respect to u. Let $\mathfrak{A} = \bar{e}\mathfrak{A}\bar{e} + \bar{e}\bar{\mathfrak{B}} + \bar{\mathfrak{B}}\bar{e} + \bar{\mathfrak{F}}$. We have $e = \bar{e} + y$, where $y = -x_2 x_3 + x_3$. Let $y = y_1 + y_2 + y_3 + y_4$, where y_1, y_2, y_3, and y_4 are respectively in $\bar{e}\mathfrak{A}\bar{e}$, $\bar{e}\bar{\mathfrak{B}}$, $\bar{\mathfrak{B}}\bar{e}$, and $\bar{\mathfrak{F}}$. Then $\bar{e}y = y_1 + y_2$. But

$$\bar{e}y = (u + x_2)(-x_2 x_3 + x_3) = 0.$$

Hence $y_1 + y_2 = 0$, and $y = y_3 + y_4$. Then $e = \bar{e} + y_3 + y_4$ is idempotent, which demands that $y_4 y_3 + y_4{}^2 - y_4 = 0$. But $y_4 y_3$ and $y_4{}^2 - y_4$ are respectively in $\bar{\mathfrak{B}}\bar{e}$ and $\bar{\mathfrak{F}}$ and must therefore vanish separately. Since y_4 is nil-

potent or zero, $y_4^2 = y_4$ requires that $y_4 = 0$, i. e., $e = \bar{e} + y_3$, where y_3 is in $\bar{\mathfrak{B}}\bar{e}$, and, by theorem 4. 2, the decomposition of $\mathfrak{A}$ with respect to e is equivalent to that with respect to $\bar{e}$, and consequently to that with respect to u.

By a similar proof we may establish the following theorem.

THEOREM 4. 4. *If u is a principal idempotent of $\mathfrak{A}$ and $u\mathfrak{B}^2u = 0$, the decomposition of $\mathfrak{A}$ with respect to any principal idempotent is equivalent to that with respect to u.*

5. *Modulus and semi-modulus.*

THEOREM 5. 1. *If u is a principal idempotent of $\mathfrak{A}$, $\mathfrak{A}$ possesses a modulus if and only if $\mathfrak{B} = 0$ in the decomposition of $\mathfrak{A}$ with respect to u.*

For, if $\mathfrak{B} = 0$, u is evidently a modulus for $\mathfrak{A}$. But if $\mathfrak{A}$ possesses a modulus u, it is the only principal idempotent of $\mathfrak{A}$ and $\mathfrak{A} = u\mathfrak{A}u$.

If $\mathfrak{A}$ possesses an element u such that $ux = x$ $(xu = x)$ for every x in $\mathfrak{A}$, but $xu \neq x$ $(ux \neq x)$ for some x in $\mathfrak{A}$, we shall call u a *left (right) hand semi-modulus of* $\mathfrak{A}$.

THEOREM 5. 2. *A semi-modulus of $\mathfrak{A}$ is a principal idempotent of $\mathfrak{A}$.*

For, if u is a semi-modulus of A and e any idempotent of $\mathfrak{A}$, either $ue = e \neq 0$ or $eu = e \neq 0$.

THEOREM 5. 3. *If $\mathfrak{A}$ has a semi-modulus, every principal idempotent of $\mathfrak{A}$ is a semi-modulus of the same type.*

For, let u be a left-hand semi-modulus of $\mathfrak{A}$. Then

$$\mathfrak{B}u + \mathfrak{F} = u(\mathfrak{B}u + \mathfrak{F}) = 0 \quad \text{and} \quad \mathfrak{A} = u\mathfrak{A}u + u\mathfrak{B}.$$

By theorem 3. 3, every principal idempotent is of the form $u + x_2$, where x_2 is in $u\mathfrak{B}$. If y is any element of $\mathfrak{A}$, we have

$$(u + x_2)y = (u + x_2)uy = uy = y.$$

A similar proof holds if u is a right-hand semi-modulus.

COROLLARY 5. 31. *If $\mathfrak{A}$ has a semi-modulus it has an infinite number of them.*

For, if u is a left-hand semi-modulus, $u\mathfrak{B} \neq 0$ and contains an infinite number of elements x. For each of these $u + x$ is a left-hand semi-modulus.

COROLLARY 5. 32. *If $\mathfrak{A}$ has a semi-modulus, the decompositions of $\mathfrak{A}$ with respect to principal idempotents are all equivalent.*

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